Classical Mechanics
An introductory course

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Contents

1 Introduction

1.1 Major sources:

The sources which I consulted most frequently whilst developing this course are:

Analytical Mechanics: G.R. Fowles, Third edition (Holt, Rinehart, & Winston, New York NY, 1977).

Physics: R. Resnick, D. Halliday, and K.S. Krane, Fourth edition, Vol. 1 (John Wiley & Sons, New York NY, 1992).

Encyclopædia Brittanica: Fifteenth edition (Encyclopædia Brittanica, Chicago IL, 1994).

Physics for scientists and engineers: R.A. Serway, and R.J. Beichner, Fifth edition, Vol. 1 (Saunders College Publishing, Orlando FL, 2000).

1.2 What is classical mechanics?

Classical mechanics is the study of the *motion* of bodies (including the special case in which bodies remain at rest) in accordance with the general principles first enunciated by Sir Isaac Newton in his *Philosophiae Naturalis Principia Mathematica* (1687), commonly known as the *Principia*. Classical mechanics was the first branch of Physics to be discovered, and is the foundation upon which all other branches of Physics are built. Moreover, classical mechanics has many important applications in other areas of science, such as Astronomy (*e.g.*, celestial mechanics), Chemistry (*e.g.*, the dynamics of molecular collisions), Geology (*e.g.*, the propagation of seismic waves, generated by earthquakes, through the Earth's crust), and Engineering (*e.g.*, the equilibrium and stability of structures). Classical mechanics is also of great significance outside the realm of science. After all, the sequence of events leading to the discovery of classical mechanics—starting with the ground-breaking work of Copernicus, continuing with the researches of Galileo, Kepler, and Descartes, and culminating in the monumental achievements of Newton—involved the complete overthrow of the Aristotelian picture of the Universe, which had previously prevailed for more than a millennium, and its replacement by a recognizably modern picture in which humankind no longer played a privileged role.

In our investigation of classical mechanics we shall study many different types of motion, including:

Translational motion—motion by which a body shifts from one point in space to another (*e.g.*, the motion of a bullet fired from a gun).

Rotational motion—motion by which an extended body changes orientation, with respect to other bodies in space, without changing position (*e.g.*, the motion of a spinning top).

Oscillatory motion—motion which continually repeats in time with a fixed period (*e.g.*, the motion of a pendulum in a grandfather clock).

Circular motion—motion by which a body executes a circular orbit about another fixed body [*e.g.*, the (approximate) motion of the Earth about the Sun].

Of course, these different types of motion can be combined: for instance, the motion of a properly bowled bowling ball consists of a combination of translational and rotational motion, whereas wave propagation is a combination of translational and oscillatory motion. Furthermore, the above mentioned types of motion are not entirely distinct: *e.g.*, circular motion contains elements of both rotational and oscillatory motion. We shall also study *statics*: *i.e.*, the subdivision of mechanics which is concerned with the forces that act on bodies *at rest* and in equilibrium. Statics is obviously of great importance in civil engineering: for instance, the principles of statics were used to design the building in which this lecture is taking place, so as to ensure that it does not collapse.

1.3 mks units

The first principle of any exact science is *measurement*. In mechanics there are three fundamental quantities which are subject to measurement:

1. Intervals in space: *i.e.*, lengths.
2. Quantities of inertia, or mass, possessed by various bodies.
3. Intervals in time.

Any other type of measurement in mechanics can be reduced to some combination of measurements of these three quantities.

Each of the three fundamental quantities—*length*, *mass*, and *time*—is measured with respect to some convenient standard. The system of units currently used by all scientists, and most engineers, is called the *mks system*—after the first initials of the names of the units of length, mass, and time, respectively, in this system: *i.e.*, the *meter*, the *kilogram*, and the *second*.

The mks unit of length is the *meter* (symbol m), which was formerly the distance between two scratches on a platinum-iridium alloy bar kept at the International Bureau of Metric Standard in Sèvres, France, but is now defined as the distance occupied by $1,650,763.73$ wavelengths of light of the orange-red spectral line of the isotope Krypton 86 in vacuum.

The mks unit of mass is the *kilogram* (symbol kg), which is defined as the mass of a platinum-iridium alloy cylinder kept at the International Bureau of Metric Standard in Sèvres, France.

The mks unit of time is the *second* (symbol s), which was formerly defined in terms of the Earth's rotation, but is now defined as the time for $9,192,631,770$ oscillations associated with the transition between the two hyperfine levels of the ground state of the isotope Cesium 133.

In addition to the three fundamental quantities, classical mechanics also deals with *derived quantities*, such as velocity, acceleration, momentum, angular momentum, *etc.* Each of these derived quantities can be reduced to some particular combination of length, mass, and time. The mks units of these derived quantities are, therefore, the corresponding combinations of the mks units of length, mass, and time. For instance, a velocity can be reduced to a length divided by a time. Hence, the mks units of velocity are meters per second:

$$[v] = \frac{[\mathrm{L}]}{[\mathrm{T}]} = \mathrm{m\,s^{-1}}. \tag{1.1}$$

Here, v stands for a velocity, L for a length, and T for a time, whereas the operator $[\cdots]$ represents the units, or *dimensions*, of the quantity contained within the brackets. Momentum can be reduced to a mass times a velocity. Hence, the mks units of momentum are kilogram-meters per second:

$$[p] = [\mathrm{M}][v] = \frac{[\mathrm{M}][\mathrm{L}]}{[\mathrm{T}]} = \mathrm{kg\,m\,s^{-1}}. \tag{1.2}$$

Here, p stands for a momentum, and M for a mass. In this manner, the mks units of all derived quantities appearing in classical dynamics can easily be obtained.

1.4 Standard prefixes

mks units are specifically designed to conveniently describe those motions which occur in everyday life. Unfortunately, mks units tend to become rather unwieldy when dealing with motions on very small scales (*e.g.*, the motions of molecules) or very large scales (*e.g.*, the motion of stars in the Galaxy). In order to help cope with this problem, a set of standard prefixes has been devised, which allow the mks units of length, mass, and time to be modified so as to deal more easily with very small and very large quantities: these prefixes are specified in Tab. 1. Thus, a *kilometer* (km) represents 10^3 m, a *nanometer* (nm) represents 10^{-9} m, and a *femtosecond* (fs) represents 10^{-15} s. The standard prefixes can also be used to modify the units of derived quantities.

Factor	*Prefix*	*Symbol*	*Factor*	*Prefix*	*Symbol*
10^{18}	exa-	E	10^{-1}	deci-	d
10^{15}	peta-	P	10^{-2}	centi-	c
10^{12}	tera-	T	10^{-3}	milli-	m
10^{9}	giga-	G	10^{-6}	micro-	μ
10^{6}	mega-	M	10^{-9}	nano-	n
10^{3}	kilo-	k	10^{-12}	pico-	p
10^{2}	hecto-	h	10^{-15}	femto-	f
10^{1}	deka-	da	10^{-18}	atto-	a

Table 1: *Standard prefixes*

1.5 Other units

The mks system is not the only system of units in existence. Unfortunately, the obsolete cgs (centimeter-gram-second) system and the even more obsolete fps (foot-pound-second) system are still in use today, although their continued employment is now *strongly discouraged* in science and engineering (except in the US!). Conversion between different systems of units is, in principle, perfectly straightforward, but, in practice, a frequent source of error. Witness, for example, the recent loss of the Mars Climate Orbiter because the engineers who designed its rocket engine used fps units whereas the NASA mission controllers employed mks units. Table 2 specifies the various conversion factors between mks, cgs, and fps units. Note that, rather confusingly (unless you are an engineer in the US!), a pound is a unit of force, rather than mass. Additional non-standard units of length include the inch ($1\,\text{ft} = 12\,\text{in}$), the yard ($1\,\text{ya} = 3\,\text{ft}$), and the mile ($1\,\text{mi} = 5,280\,\text{ft}$). Additional non-standard units of mass include the ton (in the US, $1\,\text{ton} = 2,000\,\text{lb}$; in the UK, $1\,\text{ton} = 2,240\,\text{lb}$), and the metric ton ($1\,\text{tonne} = 1,000\,\text{kg}$). Finally, additional non-standard units of time include the minute ($1\,\text{min} = 60\,\text{s}$), the hour ($1\,\text{hr} = 3,600\,\text{s}$), the day ($1\,\text{da} = 86,400\,\text{s}$), and the year ($1\,\text{yr} = 365.26\,\text{da} = 31,558,464\,\text{s}$).

1 cm	=	10^{-2} m
1 g	=	10^{-3} kg
1 ft	=	0.3048 m
1 lb	=	4.448 N
1 slug	=	14.59 kg

Table 2: *Conversion factors*

1.6 Precision and significant figures

In this course, you are expected to perform calculations to a relative accuracy of 1%: *i.e.*, to *three significant figures*. Since rounding errors tend to accumulate during lengthy calculations, the easiest way in which to achieve this accuracy is to perform all intermediate calculations to *four significant figures*, and then to round the final result down to three significant figures. If one of the quantities in your calculation turns out to the the small difference between two much larger numbers, then you may need to keep *more* than four significant figures. Incidentally, you are strongly urged to use *scientific notation* in *all* of your calculations: the use of non-scientific notation is generally a major source of error in this course. If your calculators are capable of operating in a mode in which *all* numbers (not just very small or very large numbers) are displayed in scientific form then you are advised to perform your calculations in this mode.

1.7 Dimensional analysis

As we have already mentioned, length, mass, and time are three *fundamentally different* quantities which are measured in three completely independent units. It, therefore, makes no sense for a prospective law of physics to express an equality between (say) a length and a mass. In other words, the example law

$$m = l, \tag{1.3}$$

where m is a mass and l is a length, cannot possibly be correct. One easy way of seeing that Eq. (1.3) is invalid (as a law of physics), is to note that this equation is dependent on the adopted system of units: *i.e.*, if $m = l$ in mks units, then $m \neq l$

in fps units, because the conversion factors which must be applied to the left- and right-hand sides differ. Physicists hold very strongly to the assumption that the laws of physics possess *objective reality*: in other words, the laws of physics are the same for all observers. One immediate consequence of this assumption is that a law of physics must take the same form in all possible systems of units that a prospective observer might choose to employ. The only way in which this can be the case is if all laws of physics are *dimensionally consistent*: *i.e.*, the quantities on the left- and right-hand sides of the equality sign in any given law of physics must have the same dimensions (*i.e.*, the same combinations of length, mass, and time). A dimensionally consistent equation naturally takes the same form in all possible systems of units, since the same conversion factors are applied to both sides of the equation when transforming from one system to another.

As an example, let us consider what is probably the most famous equation in physics:

$$E = m\,c^2. \tag{1.4}$$

Here, E is the energy of a body, m is its mass, and c is the velocity of light in vacuum. The dimensions of energy are $[M][L^2]/[T^2]$, and the dimensions of velocity are $[L]/[T]$. Hence, the dimensions of the left-hand side are $[M][L^2]/[T^2]$, whereas the dimensions of the right-hand side are $[M]\,([L]/[T])^2 = [M][L^2]/[T^2]$. It follows that Eq. (1.4) is indeed dimensionally consistent. Thus, $E = m\,c^2$ holds good in mks units, in cgs units, in fps units, and in any other sensible set of units. Had Einstein proposed $E = m\,c$, or $E = m\,c^3$, then his error would have been immediately apparent to other physicists, since these prospective laws are not dimensionally consistent. In fact, $E = m\,c^2$ represents the *only* simple, dimensionally consistent way of combining an energy, a mass, and the velocity of light in a law of physics.

The last comment leads naturally to the subject of *dimensional analysis*: *i.e.,* the use of the idea of dimensional consistency to *guess* the forms of simple laws of physics. It should be noted that dimensional analysis is of fairly limited applicability, and is a poor substitute for analysis employing the actual laws of physics; nevertheless, it is occasionally useful. Suppose that a special effects studio wants to film a scene in which the Leaning Tower of Pisa topples to the ground. In order to achieve this, the studio might make a scale model of the tower, which

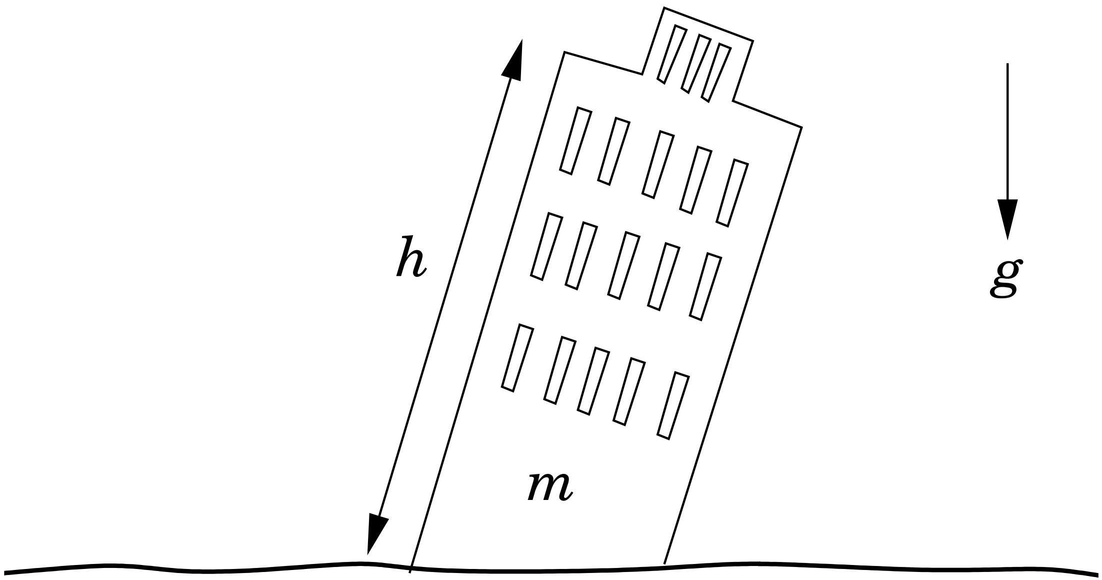


Figure 1: *The Leaning Tower of Pisa*

is (say) 1 m tall, and then film the model falling over. The only problem is that the resulting footage would look completely unrealistic, because the model tower would fall over too quickly. The studio could easily fix this problem by slowing the film down. The question is by what factor should the film be slowed down in order to make it look realistic?

Although, at this stage, we do not know how to apply the laws of physics to the problem of a tower falling over, we can, at least, make some educated guesses as to what factors the time t_f required for this process to occur depends on. In fact, it seems reasonable to suppose that t_f depends principally on the mass of the tower, m, the height of the tower, h, and the acceleration due to gravity, g. See Fig. 1. In other words,

$$t_f = C\, m^x\, h^y\, g^z, \tag{1.5}$$

where C is a dimensionless constant, and x, y, and z are unknown exponents. The exponents x, y, and z can be determined by the requirement that the above equation be dimensionally consistent. Incidentally, the dimensions of an acceleration are $[L]/[T^2]$. Hence, equating the dimensions of both sides of Eq. (1.5), we obtain

$$[T] = [M]^x\, [L]^y \left(\frac{[L]}{[T^2]}\right)^z. \tag{1.6}$$

We can now compare the exponents of [L], [M], and [T] on either side of the above expression: these exponents must all match in order for Eq. (1.5) to be dimensionally consistent. Thus,

$$0 = y + z, \tag{1.7}$$
$$0 = x, \tag{1.8}$$
$$1 = -2\,z. \tag{1.9}$$

It immediately follows that $x = 0$, $y = 1/2$, and $z = -1/2$. Hence,

$$t_f = C\,\sqrt{\frac{h}{g}}. \tag{1.10}$$

Now, the actual tower of Pisa is approximately 100 m tall. It follows that since $t_f \propto \sqrt{h}$ (g is the same for both the real and the model tower) then the 1 m high model tower falls over a factor of $\sqrt{100/1} = 10$ times faster than the real tower. Thus, the film must be slowed down by a factor 10 in order to make it look realistic.

Worked example 1.1: Conversion of units

Question: Farmer Jones has recently brought a 40 acre field and wishes to replace the fence surrounding it. Given that the field is square, what length of fencing (in meters) should Farmer Jones purchase? Incidentally, 1 acre equals 43,560 square feet.

Answer: If 1 acre equals 43,560 ft^2 and 1 ft equals 0.3048 m (see Tab. 2) then

$$1\text{ acre} = 43560 \times (0.3048)^2 = 4.047 \times 10^3\,\text{m}^2.$$

Thus, the area of the field in mks units is

$$A = 40 \times 4.047 \times 10^3 = 1.619 \times 10^5\,\text{m}^2.$$

Now, a square field with sides of length l has an area $A = l^2$ and a circumference $D = 4l$. Hence, $D = 4\sqrt{A}$. It follows that the length of the fence is

$$D = 4 \times \sqrt{1.619 \times 10^5} = 1.609 \times 10^3\,\text{m}.$$

Worked example 1.2: Tire pressure

Question: The recommended tire pressure in a Honda Civic is 28 psi (pounds per square inch). What is this pressure in atmospheres (1 atmosphere is $10^5\,\mathrm{N\,m^{-2}}$)?

Answer: First, 28 pounds per square inch is the same as $28 \times (12)^2 = 4032$ pounds per square foot (the standard fps unit of pressure). Now, 1 pound equals 4.448 Newtons (the standard SI unit of force), and 1 foot equals $0.3048\,\mathrm{m}$ (see Tab. 2). Hence,

$$P = 4032 \times (4.448)/(0.3048)^2 = 1.93 \times 10^5\,\mathrm{Nm^{-2}}.$$

It follows that 28 psi is equivalent to 1.93 atmospheres.

Worked example 1.3: Dimensional analysis

Question: The speed of sound v in a gas might plausibly depend on the pressure p, the density ρ, and the volume V of the gas. Use dimensional analysis to determine the exponents x, y, and z in the formula

$$v = C\,p^x\,\rho^y\,V^z,$$

where C is a dimensionless constant. Incidentally, the mks units of pressure are kilograms per meter per second squared.

Answer: Equating the dimensions of both sides of the above equation, we obtain

$$\frac{[L]}{[T]} = \left(\frac{[M]}{[T^2][L]}\right)^x \left(\frac{[M]}{[L^3]}\right)^y [L^3]^z.$$

A comparison of the exponents of $[L]$, $[M]$, and $[T]$ on either side of the above expression yields

$$\begin{aligned} 1 &= -x - 3y + 3z, \\ 0 &= x + y, \\ -1 &= -2x. \end{aligned}$$

The third equation immediately gives $x = 1/2$; the second equation then yields $y = -1/2$; finally, the first equation gives $z = 0$. Hence,

$$v = C\sqrt{\frac{p}{\rho}}.$$

2 Motion in 1 dimension

2.1 Introduction

The purpose of this section is to introduce the concepts of *displacement*, *velocity*, and *acceleration*. For the sake of simplicity, we shall restrict our attention to 1-dimensional motion.

2.2 Displacement

Consider a body moving in 1 dimension: *e.g.*, a train traveling down a straight railroad track, or a truck driving down an interstate in Kansas. Suppose that we have a team of observers who continually report the location of this body to us as time progresses. To be more exact, our observers report the distance x of the body from some arbitrarily chosen reference point located on the track on which it is constrained to move. This point is known as the *origin* of our coordinate system. A *positive* x value implies that the body is located x meters to the *right* of the origin, whereas a *negative* x value implies that the body is located $|x|$ meters to the *left* of the origin. Here, x is termed the *displacement* of the body from the origin. See Fig. 2. Of course, if the body is extended then our observers will have to report the displacement x of some conveniently chosen reference point on the body (*e.g.*, its centre of mass) from the origin.

Our information regarding the body's motion consists of a set of data points, each specifying the displacement x of the body at some time t. It is usually illuminating to graph these points. Figure 3 shows an example of such a graph. As is often the case, it is possible to fit the data points appearing in this graph using a relatively simple analytic curve. Indeed, the curve associated with Fig. 3 is

$$x = 1 + t + \frac{t^2}{2} - \frac{t^4}{4}. \tag{2.1}$$

2.3 Velocity

Both Fig. 3 and formula (2.1) effectively specify the location of the body whose motion we are studying as time progresses. Let us now consider how we can use this information to determine the body's instantaneous *velocity* as a function of time. The conventional definition of velocity is as follows:

> Velocity is the rate of change of displacement with time.

This definition implies that

$$v = \frac{\Delta x}{\Delta t}, \tag{2.2}$$

where v is the body's velocity at time t, and Δx is the change in displacement of the body between times t and $t + \Delta t$.

How should we choose the time interval Δt appearing in Eq. (2.2)? Obviously, in the simple case in which the body is moving with *constant* velocity, we can make Δt as large or small as we like, and it will not affect the value of v. Suppose, however, that v is constantly changing in time, as is generally the case. In this situation, Δt must be kept sufficiently small that the body's velocity does not change appreciably between times t and $t + \Delta t$. If Δt is made too large then formula (2.2) becomes invalid.

Suppose that we require a *general* expression for instantaneous velocity which is valid irrespective of how rapidly or slowly the body's velocity changes in time. We can achieve this goal by taking the limit of Eq. (2.2) as Δt approaches zero. This ensures that no matter how rapidly v varies with time, the velocity of the

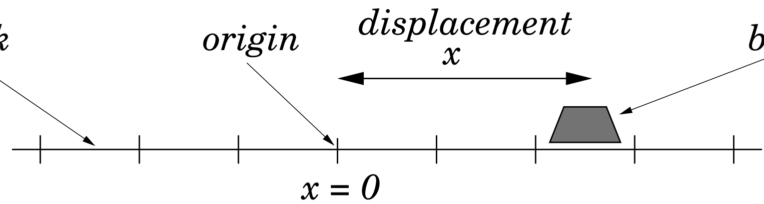


Figure 2: *Motion in 1 dimension*

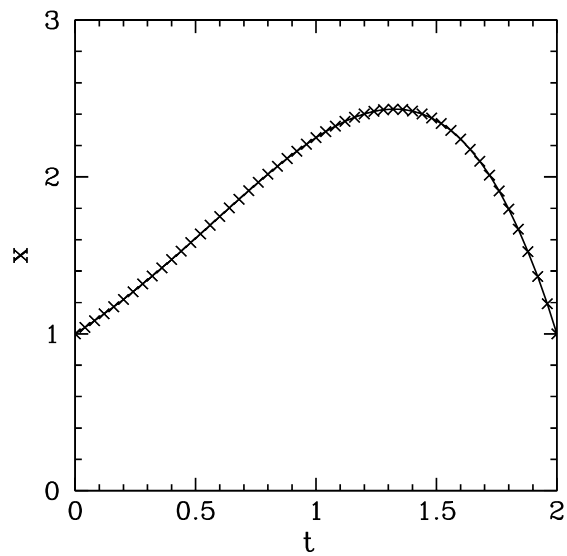


Figure 3: *Graph of displacement versus time*

body is always approximately constant in the interval t to $t + \Delta t$. Thus,

$$v = \lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t} = \frac{dx}{dt}, \tag{2.3}$$

where dx/dt represents the *derivative* of x with respect to t. The above definition is particularly useful if we can represent $x(t)$ as an analytic function, because it allows us to immediately evaluate the instantaneous velocity $v(t)$ via the rules of calculus. Thus, if $x(t)$ is given by formula (2.1) then

$$v = \frac{dx}{dt} = 1 + t - t^3. \tag{2.4}$$

Figure 4 shows the graph of v versus t obtained from the above expression. Note that when v is positive the body is moving to the right (*i.e.*, x is increasing in time). Likewise, when v is negative the body is moving to the left (*i.e.*, x is decreasing in time). Finally, when $v = 0$ the body is instantaneously at rest.

The terms velocity and speed are often confused with one another. A velocity can be either positive or negative, depending on the direction of motion. The conventional definition of *speed* is that it is the magnitude of velocity (*i.e.*, it is v with the sign stripped off). It follows that a body can never possess a negative speed.

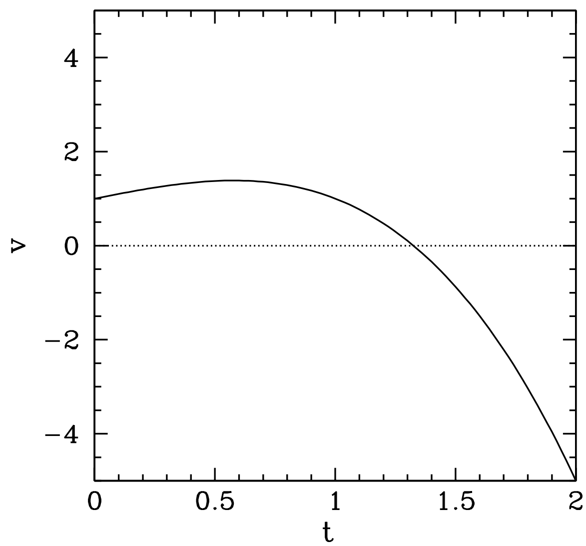


Figure 4: *Graph of instantaneous velocity versus time associated with the motion specified in Fig. 3*

2.4 Acceleration

The conventional definition of acceleration is as follows:

> Acceleration is the rate of change of velocity with time.

This definition implies that

$$a = \frac{\Delta v}{\Delta t}, \tag{2.5}$$

where a is the body's acceleration at time t, and Δv is the change in velocity of the body between times t and $t + \Delta t$.

How should we choose the time interval Δt appearing in Eq. (2.5)? Again, in the simple case in which the body is moving with *constant* acceleration, we can make Δt as large or small as we like, and it will not affect the value of a. Suppose, however, that a is constantly changing in time, as is generally the case. In this situation, Δt must be kept sufficiently small that the body's acceleration does not change appreciably between times t and $t + \Delta t$.

A general expression for instantaneous acceleration, which is valid irrespective of how rapidly or slowly the body's acceleration changes in time, can be obtained

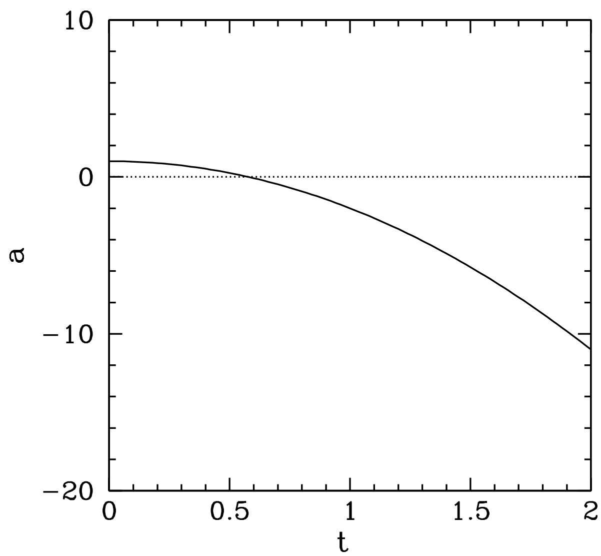


Figure 5: *Graph of instantaneous acceleration versus time associated with the motion specified in Fig. 3*

by taking the limit of Eq. (2.5) as Δt approaches zero:

$$a = \lim_{\Delta t \to 0} \frac{\Delta v}{\Delta t} = \frac{dv}{dt} = \frac{d^2x}{dt^2}. \tag{2.6}$$

The above definition is particularly useful if we can represent $x(t)$ as an analytic function, because it allows us to immediately evaluate the instantaneous acceleration $a(t)$ via the rules of calculus. Thus, if $x(t)$ is given by formula (2.1) then

$$a = \frac{d^2x}{dt^2} = 1 - 3t^2. \tag{2.7}$$

Figure 5 shows the graph of a versus time obtained from the above expression. Note that when a is positive the body is *accelerating* to the right (*i.e.*, v is increasing in time). Likewise, when a is negative the body is *decelerating* (*i.e.*, v is decreasing in time).

Fortunately, it is generally not necessary to evaluate the rate of change of acceleration with time, since this quantity does not appear in Newton's laws of motion.

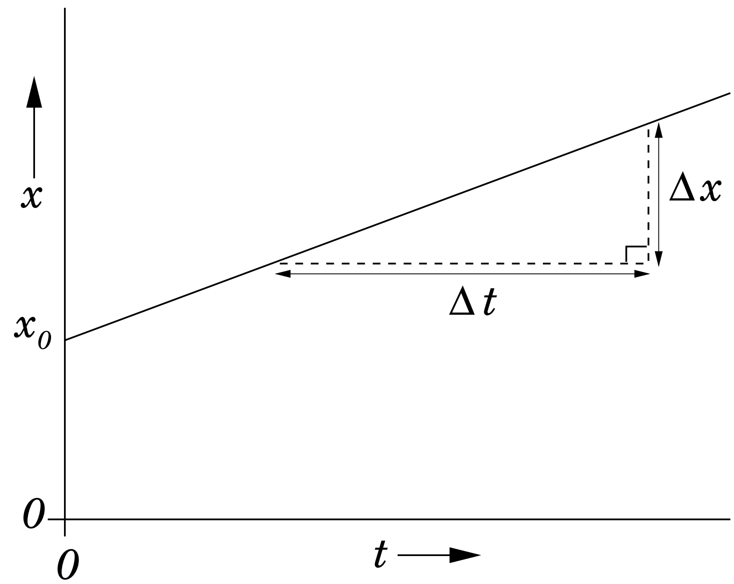


Figure 6: *Graph of displacement versus time for a body moving with constant velocity*

2.5 Motion with constant velocity

The simplest type of motion (excluding the trivial case in which the body under investigation remains at rest) consists of motion with *constant velocity*. This type of motion occurs in everyday life whenever an object slides over a horizontal, low friction surface: *e.g.*, a puck sliding across a hockey rink.

Fig. 6 shows the graph of displacement versus time for a body moving with constant velocity. It can be seen that the graph consists of a *straight-line*. This line can be represented algebraically as

$$x = x_0 + v\,t. \tag{2.8}$$

Here, x_0 is the displacement at time $t = 0$: this quantity can be determined from the graph as the *intercept* of the straight-line with the x-axis. Likewise, $v = dx/dt$ is the constant velocity of the body: this quantity can be determined from the graph as the *gradient* of the straight-line (*i.e.*, the ratio $\Delta x/\Delta t$, as shown). Note that $a = d^2x/dt^2 = 0$, as expected.

Fig. 7 shows a displacement versus time graph for a slightly more complicated case of motion with constant velocity. The body in question moves to the right

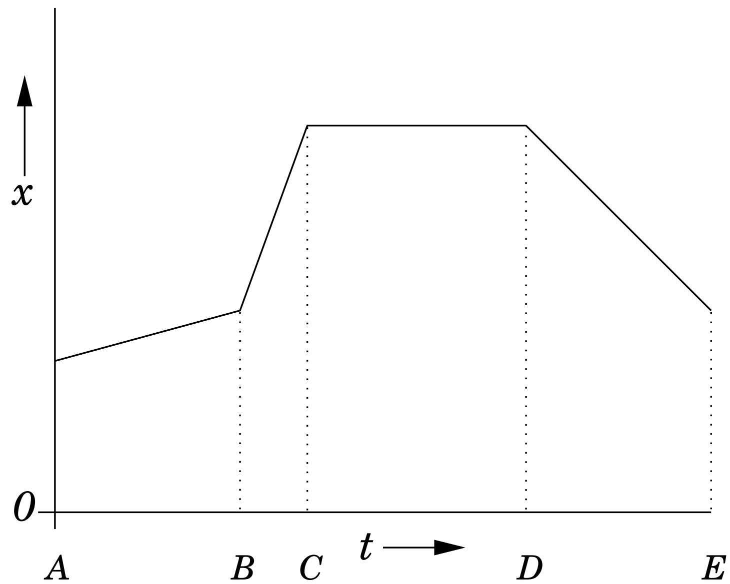


Figure 7: *Graph of displacement versus time*

(since x is clearly increasing with t) with a constant velocity (since the graph is a straight-line) between times A and B. The body then moves to the right (since x is still increasing in time) with a somewhat larger constant velocity (since the graph is again a straight line, but possesses a larger gradient than before) between times B and C. The body remains at rest (since the graph is horizontal) between times C and D. Finally, the body moves to the left (since x is decreasing with t) with a constant velocity (since the graph is a straight-line) between times D and E.

2.6 Motion with constant acceleration

Motion with constant acceleration occurs in everyday life whenever an object is dropped: the object moves downward with the constant acceleration $9.81\,\mathrm{m\,s^{-2}}$, under the influence of gravity.

Fig. 8 shows the graphs of displacement versus time and velocity versus time for a body moving with constant acceleration. It can be seen that the displacement-time graph consists of a *curved-line* whose gradient (slope) is increasing in time.

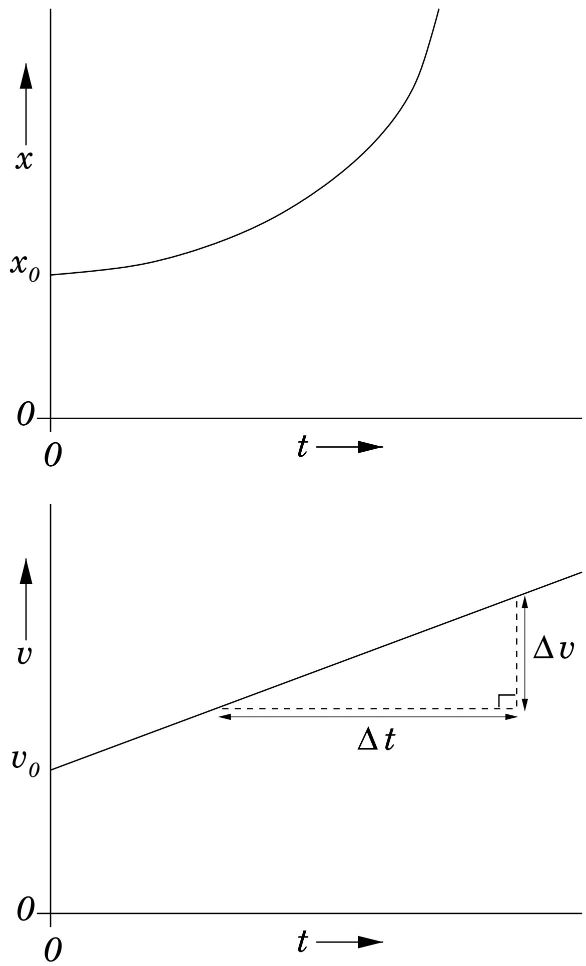


Figure 8: *Graphs of displacement versus time and velocity versus time for a body moving with constant acceleration*

This line can be represented algebraically as

$$x = x_0 + v_0\,t + \frac{1}{2}\,a\,t^2. \tag{2.9}$$

Here, x_0 is the displacement at time $t = 0$: this quantity can be determined from the graph as the *intercept* of the curved-line with the x-axis. Likewise, v_0 is the body's instantaneous velocity at time $t = 0$.

The velocity-time graph consists of a *straight-line* which can be represented algebraically as

$$v = \frac{dx}{dt} = v_0 + a\,t. \tag{2.10}$$

The quantity v_0 is determined from the graph as the *intercept* of the straight-line with the x-axis. The quantity a is the constant acceleration: this can be determined graphically as the *gradient* of the straight-line (*i.e.*, the ratio $\Delta v/\Delta t$, as shown). Note that $dv/dt = a$, as expected.

Equations (2.9) and (2.10) can be rearranged to give the following set of three useful formulae which characterize motion with constant acceleration:

$$s = v_0\,t + \frac{1}{2}\,a\,t^2, \tag{2.11}$$

$$v = v_0 + a\,t, \tag{2.12}$$

$$v^2 = v_0^2 + 2\,a\,s. \tag{2.13}$$

Here, $s = x - x_0$ is the net distance traveled after t seconds.

Fig. 9 shows a displacement versus time graph for a slightly more complicated case of accelerated motion. The body in question accelerates to the right [since the gradient (slope) of the graph is increasing in time] between times A and B. The body then moves to the right (since x is increasing in time) with a constant velocity (since the graph is a straight line) between times B and C. Finally, the body decelerates [since the gradient (slope) of the graph is decreasing in time] between times C and D.

2.7 Free-fall under gravity

Galileo Galilei was the first scientist to appreciate that, *neglecting the effect of air resistance*, all bodies in free-fall close to the Earth's surface accelerate vertically downwards with the same acceleration: namely, $g = 9.81\,\mathrm{m\,s^{-2}}$.[1] The neglect of air resistance is a fairly good approximation for large objects which travel relatively slowly (*e.g.*, a shot-putt, or a basketball), but becomes a poor approximation for small objects which travel relatively rapidly (*e.g.*, a golf-ball, or a bullet fired from a pistol).

[1] Actually, the acceleration due to gravity varies slightly over the Earth's surface because of the combined effects of the Earth's rotation and the Earth's slightly flattened shape. The acceleration at the poles is about $9.834\,\mathrm{m\,s^{-2}}$, whereas the acceleration at the equator is only $9.780\,\mathrm{m\,s^{-2}}$. The *average* acceleration is $9.81\,\mathrm{m\,s^{-2}}$.

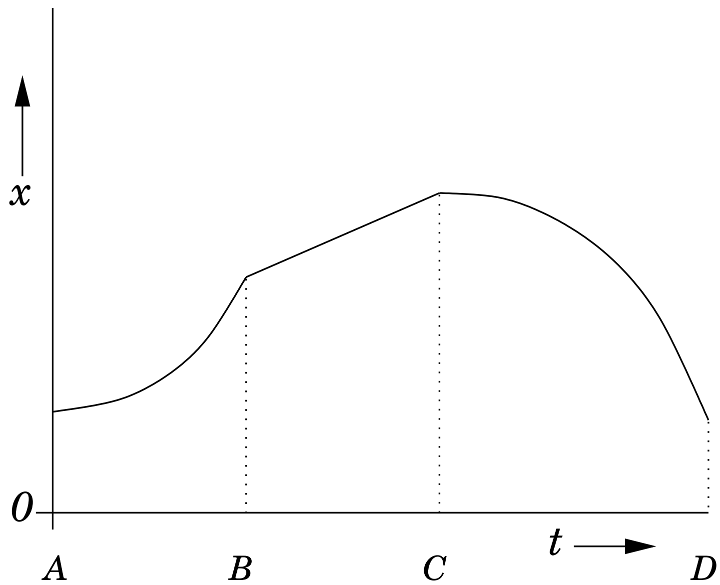


Figure 9: *Graph of displacement versus time*

Equations (2.11)–(2.13) can easily be modified to deal with the special case of an object free-falling under gravity:

$$s = v_0\,t - \frac{1}{2}\,g\,t^2, \tag{2.14}$$

$$v = v_0 - g\,t, \tag{2.15}$$

$$v^2 = v_0^2 - 2\,g\,s. \tag{2.16}$$

Here, $g = 9.81\,\mathrm{m\,s^{-2}}$ is the downward acceleration due to gravity, s is the distance the object has moved vertically between times $t = 0$ and t (if $s > 0$ then the object has risen s meters, else if $s < 0$ then the object has fallen $|s|$ meters), and v_0 is the object's instantaneous velocity at $t = 0$. Finally, v is the object's instantaneous velocity at time t.

Let us illustrate the use of Eqs. (2.14)–(2.16). Suppose that a ball is released from rest and allowed to fall under the influence of gravity. How long does it take the ball to fall h meters? Well, according to Eq. (2.14) [with $v_0 = 0$ (since the ball is released from rest), and $s = -h$ (since we wish the ball to *fall* h meters)], $h = g\,t^2/2$, so the time of fall is

$$t = \sqrt{\frac{2\,h}{g}}. \tag{2.17}$$

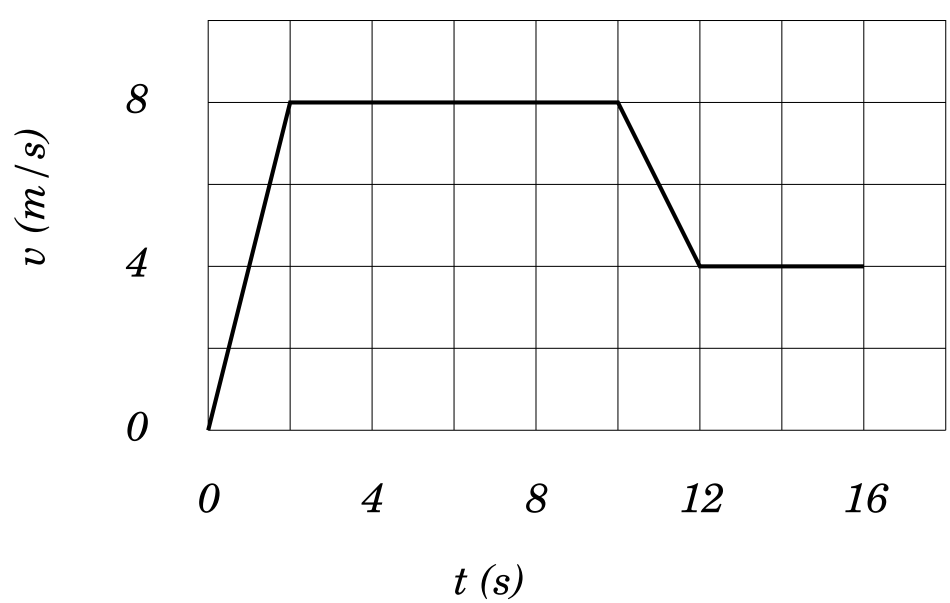
Suppose that a ball is thrown vertically upwards from ground level with velocity u. To what height does the ball rise, how long does it remain in the air, and with what velocity does it strike the ground? The ball attains its maximum height when it is momentarily at rest (*i.e.*, when $v = 0$). According to Eq. (2.15) (with $v_0 = u$), this occurs at time $t = u/g$. It follows from Eq. (2.14) (with $v_0 = u$, and $t = u/g$) that the maximum height of the ball is given by

$$h = \frac{u^2}{2\,g}. \tag{2.18}$$

When the ball strikes the ground it has traveled zero net meters vertically, so $s = 0$. It follows from Eqs. (2.15) and (2.16) (with $v_0 = u$ and $t > 0$) that $v = -u$. In other words, the ball hits the ground with an equal and opposite velocity to that with which it was thrown into the air. Since the ascent and decent phases of the ball's trajectory are clearly symmetric, the ball's time of flight is simply twice the time required for the ball to attain its maximum height: *i.e.*,

$$t = \frac{2\,u}{g}. \tag{2.19}$$

Worked example 2.1: Velocity-time graph



Question: Consider the motion of the object whose velocity-time graph is given in the diagram.

1. What is the acceleration of the object between times $t = 0$ and $t = 2$?
2. What is the acceleration of the object between times $t = 10$ and $t = 12$?
3. What is the net displacement of the object between times $t = 0$ and $t = 16$?

Answer:

1. The v-t graph is a straight-line between $t = 0$ and $t = 2$, indicating constant acceleration during this time period. Hence,
$$a = \frac{\Delta v}{\Delta t} = \frac{v(t = 2) - v(t = 0)}{2 - 0} = \frac{8 - 0}{2} = 4\,\mathrm{m\,s^{-2}}.$$
2. The v-t graph is a straight-line between $t = 10$ and $t = 12$, indicating constant acceleration during this time period. Hence,
$$a = \frac{\Delta v}{\Delta t} = \frac{v(t = 12) - v(t = 10)}{12 - 10} = \frac{4 - 8}{2} = -2\,\mathrm{m\,s^{-2}}.$$
The negative sign indicates that the object is decelerating.
3. Now, $v = dx/dt$, so
$$x(16) - x(0) = \int_0^{16} v(t)\,dt.$$
In other words, the net displacement between times $t = 0$ and $t = 16$ equals the area under the v-t curve, evaluated between these two times. Recalling that the area of a triangle is half its width times its height, the number of grid-squares under the v-t curve is 25. The area of each grid-square is $2 \times 2 = 4$ m. Hence,
$$x(16) - x(0) = 4 \times 25 = 100\,\mathrm{m}.$$

Worked example 2.2: Speed trap

Question: In a speed trap, two pressure-activated strips are placed 120 m apart on a highway on which the speed limit is 85 km/h. A driver going 110 km/h notices a police car just as he/she activates the first strip, and immediately slows down. What deceleration is needed so that the car's average speed is within the speed limit when the car crosses the second strip?

Answer: Let $v_1 = 110\,\mathrm{km/h}$ be the speed of the car at the first strip. Let $\Delta x = 120\,\mathrm{m}$ be the distance between the two strips, and let Δt be the time taken by the car to travel from one strip to the other. The average velocity of the car is

$$\bar{v} = \frac{\Delta x}{\Delta t}.$$

We need this velocity to be 85 km/h. Hence, we require

$$\Delta t = \frac{\Delta x}{\bar{v}} = \frac{120}{85 \times (1000/3600)} = 5.082\,\mathrm{s}.$$

Here, we have changed units from km/h to m/s. Now, assuming that the acceleration a of the car is uniform, we have

$$\Delta x = v_1\,\Delta t + \frac{1}{2}\,a\,(\Delta t)^2,$$

which can be rearranged to give

$$a = \frac{2\,(\Delta x - v_1\,\Delta t)}{(\Delta t)^2} = \frac{2\,(120 - 110 \times (1000/3600) \times 5.082)}{(5.082)^2} = -2.73\,\mathrm{m\,s^{-2}}.$$

Hence, the required deceleration is $2.73\,\mathrm{m\,s^{-2}}$.

Worked example 2.3: The Brooklyn bridge

Question: In 1886, Steve Brodie achieved notoriety by allegedly jumping off the recently completed Brooklyn bridge, for a bet, and surviving. Given that the bridge rises 135 ft over the East River, how long would Mr. Brodie have been in the air, and with what speed would he have struck the water? Give all answers in mks units. You may neglect air resistance.

Answer: Mr. Brodie's net vertical displacement was $h = -135 \times 0.3048 = -41.15$ m. Assuming that his initial velocity was zero,

$$h = -\frac{1}{2}\,g\,t^2,$$

where t was his time of flight. Hence,

$$t = \sqrt{\frac{-2\,h}{g}} = \sqrt{\frac{2 \times 41.15}{9.81}} = 2.896\,\text{s}.$$

His final velocity was

$$v = -g\,t = -9.81 \times 2.896 = -28.41\,\text{m}\,\text{s}^{-1}.$$

Thus, the speed with which he plunged into the East River was $28.41\,\text{m}\,\text{s}^{-1}$, or 63.6 mi/h ! Clearly, Mr. Brodie's story should be taken with a pinch of salt.

3 Motion in 3 dimensions

3.1 Introduction

The purpose of this section is to generalize the previously introduced concepts of *displacement*, *velocity*, and *acceleration* in order to deal with motion in 3 dimensions.

3.2 Cartesian coordinates

Our first task, when dealing with 3-dimensional motion, is to set up a suitable coordinate system. The most straight-forward type of coordinate system is called a *Cartesian system*, after René Descartes. A Cartesian coordinate system consists of three mutually perpendicular axes, the x-, y-, and z-axes (say). By convention, the orientation of these axes is such that when the index finger, the middle finger, and the thumb of the right-hand are configured so as to be mutually perpendicular, the index finger, the middle finger, and the thumb can be aligned along the x-, y-, and z-axes, respectively. Such a coordinate system is termed *right-handed*. See Fig. 10. The point of intersection of the three coordinate axes is termed the *origin* of the coordinate system.

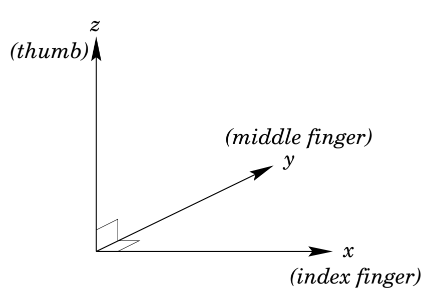


Figure 10: *A right-handed Cartesian coordinate system*

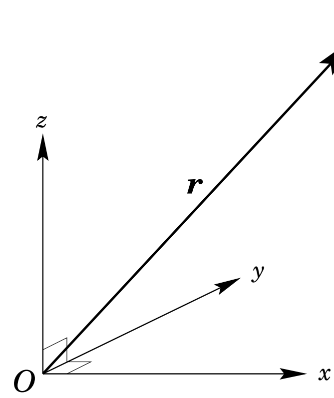


Figure 11: *A vector displacement*

3.3 Vector displacement

Consider the motion of a body moving in 3 dimensions. The body's instantaneous position is most conveniently specified by giving its displacement from the origin of our coordinate system. Note, however, that in 3 dimensions such a displacement possesses both *magnitude* and *direction*. In other words, we not only have to specify how far the body is situated from the origin, we also have to specify in which direction it lies. A quantity which possesses both magnitude and direction is termed a *vector*. By contrast, a quantity which possesses only magnitude is termed a *scalar*. Mass and time are scalar quantities. However, in general, displacement is a vector.

The vector displacement $\mathbf{r}$ of some point R from the origin O can be visualized as an arrow running from point O to point R. See Fig. 11. Note that in typeset documents vector quantities are conventionally written in a *bold-faced* font (*e.g.*, $\mathbf{r}$) to distinguish them from scalar quantities. In free-hand notation, vectors are usually *under-lined* (*e.g.*, $\underline{\mathrm{r}}$).

The vector displacement $\mathbf{r}$ can also be specified in terms of its *coordinates*:

$$\mathbf{r} = (x, y, z). \tag{3.1}$$

The above expression is interpreted as follows: in order to get from point O to point R, first move x meters along the x-axis (perpendicular to both the y- and z-axes), then move y meters along the y-axis (perpendicular to both the x- and

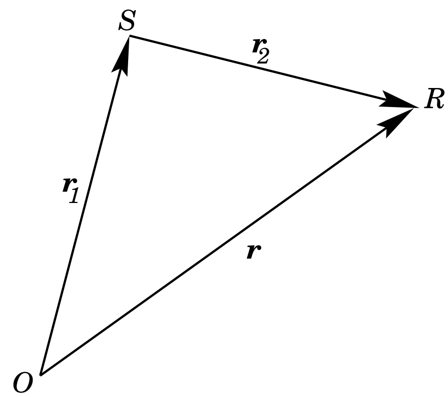


Figure 12: *Vector addition*

z-axes), finally move z meters along the z-axis (perpendicular to both the x- and y-axes). Note that a *positive* x value is interpreted as an instruction to move x meters along the x-axis in the direction of *increasing* x, whereas a *negative* x value is interpreted as an instruction to move $|x|$ meters along the x-axis in the opposite direction, and so on.

3.4 Vector addition

Suppose that the vector displacement $\mathbf{r}$ of some point R from the origin O is specified as follows:

$$\mathbf{r} = \mathbf{r}_1 + \mathbf{r}_2. \tag{3.2}$$

Figure 12 illustrates how this expression is interpreted diagrammatically: in order to get from point O to point R, we first move from point O to point S along vector $\mathbf{r}_1$, and we then move from point S to point R along vector $\mathbf{r}_2$. The net result is the same as if we had moved from point O directly to point R along vector $\mathbf{r}$. Vector $\mathbf{r}$ is termed the *resultant* of adding vectors $\mathbf{r}_1$ and $\mathbf{r}_2$.

Note that we have two ways of specifying the vector displacement of point S from the origin: we can either write $\mathbf{r}_1$ or $\mathbf{r} - \mathbf{r}_2$. The expression $\mathbf{r} - \mathbf{r}_2$ is interpreted as follows: starting at the origin, move along vector $\mathbf{r}$ in the direction of the arrow, then move along vector $\mathbf{r}_2$ in the *opposite* direction to the arrow. In other words, a *minus* sign in front of a vector indicates that we should move along that vector in the *opposite* direction to its arrow.

Suppose that the components of vectors $\mathbf{r}_1$ and $\mathbf{r}_2$ are (x_1, y_1, z_1) and (x_2, y_2, z_2), respectively. As is easily demonstrated, the components (x, y, z) of the resultant vector $\mathbf{r} = \mathbf{r}_1 + \mathbf{r}_2$ are

$$x = x_1 + x_2, \tag{3.3}$$
$$y = y_1 + y_2, \tag{3.4}$$
$$z = z_1 + z_2. \tag{3.5}$$

In other words, the components of the sum of two vectors are simply the algebraic sums of the components of the individual vectors.

3.5 Vector magnitude

If $\mathbf{r} = (x, y, z)$ represents the vector displacement of point R from the origin, what is the distance between these two points? In other words, what is the length, or *magnitude*, $r = |\mathbf{r}|$, of vector $\mathbf{r}$. It follows from a 3-dimensional generalization of Pythagoras' theorem that

$$r = \sqrt{x^2 + y^2 + z^2}. \tag{3.6}$$

Note that if $\mathbf{r} = \mathbf{r}_1 + \mathbf{r}_2$ then

$$|\mathbf{r}| \leq |\mathbf{r}_1| + |\mathbf{r}_2|. \tag{3.7}$$

In other words, the magnitudes of vectors cannot, in general, be added algebraically. The only exception to this rule (represented by the equality sign in the above expression) occurs when the vectors in question all point in the same direction. According to inequality (3.7), if we move 1 m to the North (say) and next move 1 m to the West (say) then, although we have moved a total distance of 2 m, our net distance from the starting point is less than 2 m—of course, this is just common sense.

3.6 Scalar multiplication

Suppose that $\mathbf{s} = \lambda\,\mathbf{r}$. This expression is interpreted as follows: vector $\mathbf{s}$ points in the *same* direction as vector $\mathbf{r}$, but the length of the former vector is λ times

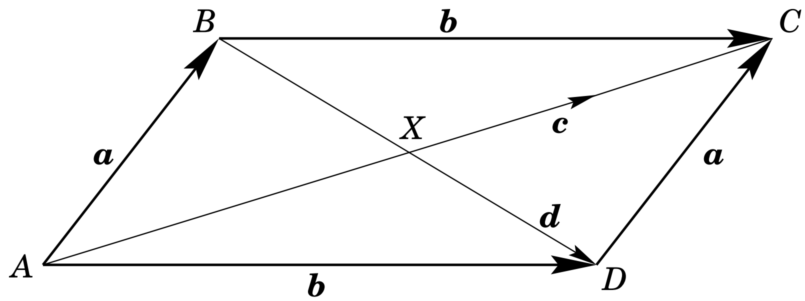


Figure 13: *A parallelogram*

that of the latter. Note that if λ is negative then vector $\mathbf{s}$ points in the *opposite* direction to vector $\mathbf{r}$, and the length of the former vector is $|\lambda|$ times that of the latter. In terms of components:

$$\mathbf{s} = \lambda\,(x, y, z) = (\lambda\,x, \lambda\,y, \lambda\,z). \tag{3.8}$$

In other words, when we multiply a vector by a scalar then the components of the resultant vector are obtained by multiplying *all* the components of the original vector by the scalar.

3.7 Diagonals of a parallelogram

The use of vectors is very well illustrated by the following rather famous proof that the diagonals of a parallelogram mutually bisect one another.

Suppose that the quadrilateral ABCD in Fig. 13 is a parallelogram. It follows that the opposite sides of ABCD can be represented by the *same* vectors, **a** and **b**: this merely indicates that these sides are of equal length and are parallel (*i.e.*, they point in the same direction). Note that Fig. 13 illustrates an important point regarding vectors. Although vectors possess both a magnitude (length) and a direction, they possess no intrinsic position information. Thus, since sides AB and DC are parallel and of equal length, they can be represented by the *same* vector **a**, despite the fact that they are in different places on the diagram.

The diagonal BD in Fig. 13 can be represented vectorially as $\mathbf{d} = \mathbf{b} - \mathbf{a}$. Likewise, the diagonal AC can be written $\mathbf{c} = \mathbf{a} + \mathbf{b}$. The displacement $\mathbf{x}$ (say) of the centroid X from point A can be written in one of two different ways:

$$\mathbf{x} = \mathbf{a} + \lambda\,\mathbf{d}, \tag{3.9}$$

$$\mathbf{x} = \mathbf{b} + \mathbf{a} - \mu\,\mathbf{c}. \tag{3.10}$$

Equation (3.9) is interpreted as follows: in order to get from point A to point X, first move to point B (along vector **a**), then move along diagonal BD (along vector **d**) for an unknown fraction λ of its length. Equation (3.10) is interpreted as follows: in order to get from point A to point X, first move to point D (along vector **b**), then move to point C (along vector **a**), finally move along diagonal CA (along vector $-\mathbf{c}$) for an unknown fraction μ of its length. Since X represents the *same* point in Eqs. (3.9) and (3.10), we can equate these two expressions to give

$$\mathbf{a} + \lambda\,(\mathbf{b} - \mathbf{a}) = \mathbf{b} + \mathbf{a} - \mu\,(\mathbf{a} + \mathbf{b}). \tag{3.11}$$

Now vectors **a** and **b** point in *different* directions, so the only way in which the above expression can be satisfied, in general, is if the coefficients of **a** and **b** match on either side of the equality sign. Thus, equating coefficients of **a** and **b**, we obtain

$$1 - \lambda = 1 - \mu, \tag{3.12}$$

$$\lambda = 1 - \mu. \tag{3.13}$$

It follows that $\lambda = \mu = 1/2$. In other words, the centroid X is located at the halfway points of diagonals BD and AC: *i.e.*, the diagonals mutually bisect one another.

3.8 Vector velocity and vector acceleration

Consider a body moving in 3 dimensions. Suppose that we know the Cartesian coordinates, x, y, and z, of this body as time, t, progresses. Let us consider how we can use this information to determine the body's instantaneous velocity and acceleration as functions of time.

The vector displacement of the body is given by

$$\mathbf{r}(t) = [x(t), y(t), z(t)]. \tag{3.14}$$

By analogy with the 1-dimensional equation (2.3), the body's vector velocity $\mathbf{v} = (v_x, v_y, v_z)$ is simply the *derivative* of $\mathbf{r}$ with respect to t. In other words,

$$\mathbf{v}(t) = \lim_{\Delta t \to 0} \frac{\mathbf{r}(t + \Delta t) - \mathbf{r}(t)}{\Delta t} = \frac{d\mathbf{r}}{dt}. \tag{3.15}$$

When written in component form, the above definition yields

$$v_x = \frac{dx}{dt}, \tag{3.16}$$

$$v_y = \frac{dy}{dt}, \tag{3.17}$$

$$v_z = \frac{dz}{dt}. \tag{3.18}$$

Thus, the x-component of velocity is simply the time derivative of the x-coordinate, and so on.

By analogy with the 1-dimensional equation (2.6), the body's vector acceleration $\mathbf{a} = (a_x, a_y, a_z)$ is simply the *derivative* of $\mathbf{v}$ with respect to t. In other words,

$$\mathbf{a}(t) = \lim_{\Delta t \to 0} \frac{\mathbf{v}(t + \Delta t) - \mathbf{v}(t)}{\Delta t} = \frac{d\mathbf{v}}{dt} = \frac{d^2\mathbf{r}}{dt^2}. \tag{3.19}$$

When written in component form, the above definition yields

$$a_x = \frac{dv_x}{dt} = \frac{d^2x}{dt^2}, \tag{3.20}$$

$$a_y = \frac{dv_y}{dt} = \frac{d^2y}{dt^2}, \tag{3.21}$$

$$a_z = \frac{dv_z}{dt} = \frac{d^2z}{dt^2}. \tag{3.22}$$

Thus, the x-component of acceleration is simply the time derivative of the x-component of velocity, and so on.

As an example, suppose that the coordinates of the body are given by

$$x = \sin t, \tag{3.23}$$

$$y = \cos t, \tag{3.24}$$

$$z = 3\,t. \tag{3.25}$$

The corresponding components of the body's velocity are then simply

$$v_x = \frac{dx}{dt} = \cos t, \tag{3.26}$$

$$v_y = \frac{dy}{dt} = -\sin t, \tag{3.27}$$

$$v_z = \frac{dz}{dt} = 3, \tag{3.28}$$

whilst the components of the body's acceleration are given by

$$a_x = \frac{dv_x}{dt} = -\sin t, \tag{3.29}$$

$$a_y = \frac{dv_y}{dt} = -\cos t, \tag{3.30}$$

$$a_z = \frac{dv_z}{dt} = 0. \tag{3.31}$$

3.9 Motion with constant velocity

An object moving in 3 dimensions with constant velocity **v** possesses a vector displacement of the form

$$\mathbf{r}(t) = \mathbf{r}_0 + \mathbf{v}\, t, \tag{3.32}$$

where the constant vector $\mathbf{r}_0$ is the displacement at time $t = 0$. Note that $d\mathbf{r}/dt = \mathbf{v}$ and $d^2\mathbf{r}/dt^2 = \mathbf{0}$, as expected. As illustrated in Fig. 14, the object's trajectory is a *straight-line* which passes through point $\mathbf{r}_0$ at time $t = 0$ and runs parallel to vector **v**.

3.10 Motion with constant acceleration

An object moving in 3 dimensions with constant acceleration **a** possesses a vector displacement of the form

$$\mathbf{r}(t) = \mathbf{r}_0 + \mathbf{v}_0\, t + \frac{1}{2}\,\mathbf{a}\, t^2. \tag{3.33}$$

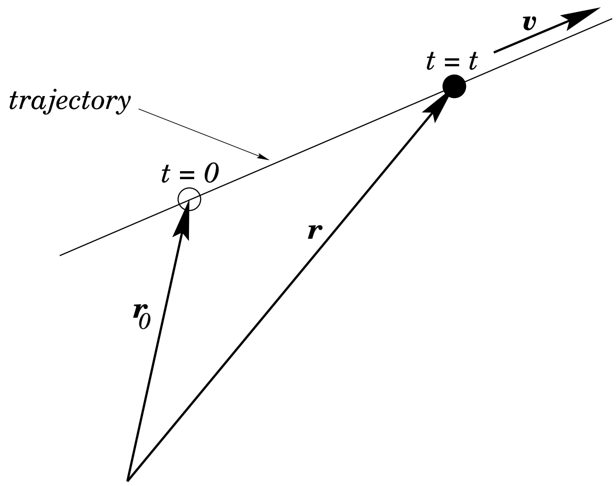


Figure 14: *Motion with constant velocity*

Hence, the object's velocity is given by

$$\mathbf{v}(t) = \frac{d\mathbf{r}}{dt} = \mathbf{v}_0 + \mathbf{a}\,t. \tag{3.34}$$

Note that $d\mathbf{v}/dt = \mathbf{a}$, as expected. In the above, the constant vectors $\mathbf{r}_0$ and $\mathbf{v}_0$ are the object's displacement and velocity at time $t = 0$, respectively.

As is easily demonstrated, the vector equivalents of Eqs. (2.11)–(2.13) are:

$$\mathbf{s} = \mathbf{v}_0\,t + \frac{1}{2}\,\mathbf{a}\,t^2, \tag{3.35}$$

$$\mathbf{v} = \mathbf{v}_0 + \mathbf{a}\,t, \tag{3.36}$$

$$v^2 = v_0^2 + 2\,\mathbf{a}\cdot\mathbf{s}. \tag{3.37}$$

These equation fully characterize 3-dimensional motion with constant acceleration. Here, $\mathbf{s} = \mathbf{r} - \mathbf{r}_0$ is the net displacement of the object between times $t = 0$ and t.

The quantity $\mathbf{a}\cdot\mathbf{s}$, appearing in Eq. (3.37), is termed the *scalar product* of vectors $\mathbf{a}$ and $\mathbf{s}$, and is defined

$$\mathbf{a}\cdot\mathbf{s} = a_x\,s_x + a_y\,s_y + a_z\,s_z. \tag{3.38}$$

The above formula has a simple geometric interpretation, which is illustrated in Fig. 15. If $|\mathbf{a}|$ is the magnitude (or length) of vector $\mathbf{a}$, $|\mathbf{s}|$ is the magnitude of

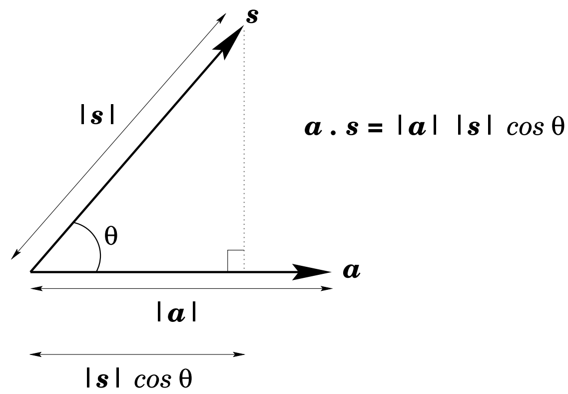


Figure 15: *The scalar product*

vector **s**, and θ is the angle subtended between these two vectors, then

$$\mathbf{a}\cdot\mathbf{s} = |\mathbf{a}|\,|\mathbf{s}|\,\cos\theta. \tag{3.39}$$

In other words, the scalar product of vectors **a** and **s** equals the product of the length of vector **a** times the length of that component of vector **s** which lies in the *same direction* as vector **a**. It immediately follows that if two vectors are mutually perpendicular (*i.e.*, $\theta = 90^\circ$) then their scalar product is zero. Furthermore, the scalar product of a vector with itself is simply the magnitude squared of that vector [this is immediately apparent from Eq. (3.38)]:

$$\mathbf{a}\cdot\mathbf{a} = |\mathbf{a}|^2 = a^2. \tag{3.40}$$

It is also apparent from Eq. (3.38) that $\mathbf{a}\cdot\mathbf{s} = \mathbf{s}\cdot\mathbf{a}$, and $\mathbf{a}\cdot(\mathbf{b}+\mathbf{c}) = \mathbf{a}\cdot\mathbf{b}+\mathbf{a}\cdot\mathbf{c}$, and $\mathbf{a}\cdot(\lambda\mathbf{s}) = \lambda(\mathbf{a}\cdot\mathbf{s})$.

Incidentally, Eq. (3.37) is obtained by taking the scalar product of Eq. (3.36) with itself, taking the scalar product of Eq. (3.35) with **a**, and then eliminating t.

3.11 Projectile motion

As a simple illustration of the concepts introduced in the previous subsections, let us examine the following problem. Suppose that a projectile is launched upward from ground level, with speed v_0, making an angle θ with the horizontal. *Neglecting the effect of air resistance*, what is the subsequent trajectory of the projectile?

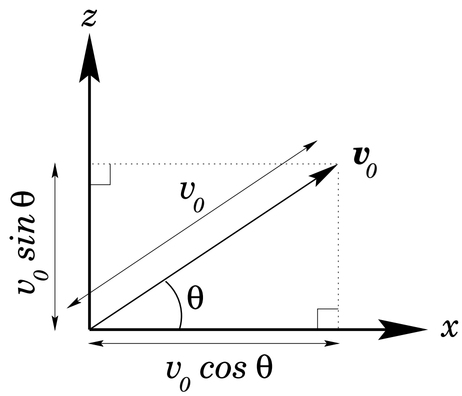


Figure 16: *Coordinates for the projectile problem*

Our first task is to set up a suitable Cartesian coordinate system. A convenient system is illustrated in Fig. 16. The z-axis points vertically upwards (this is a standard convention), whereas the x-axis points along the projectile's initial direction of horizontal motion. Furthermore, the origin of our coordinate system corresponds to the launch point. Thus, $z = 0$ corresponds to ground level.

Neglecting air resistance, the projectile is subject to a constant acceleration $g = 9.81\,\mathrm{m\,s^{-1}}$, due to gravity, which is directed *vertically downwards*. Thus, the projectile's vector acceleration is written

$$\mathbf{a} = (0, 0, -g). \tag{3.41}$$

Here, the minus sign indicates that the acceleration is in the minus z-direction (*i.e.*, downwards), as opposed to the plus z-direction (*i.e.*, upwards).

What is the initial vector velocity $\mathbf{v}_0$ with which the projectile is launched into the air at (say) $t = 0$? As illustrated in Fig. 16, given that the magnitude of this velocity is v_0, its horizontal component is directed along the x-axis, and its direction subtends an angle θ with this axis, the components of $\mathbf{v}_0$ take the form

$$\mathbf{v}_0 = (v_0 \cos\theta,\ 0,\ v_0 \sin\theta). \tag{3.42}$$

Note that $\mathbf{v}_0$ has zero component along the y-axis, which points *into* the paper in Fig. 16.

Since the projectile moves with constant acceleration, its vector displacement

$\mathbf{s} = (x, y, z)$ from its launch point satisfies [see Eq. (3.35)]

$$\mathbf{s} = \mathbf{v}_0\, t + \frac{1}{2}\,\mathbf{a}\, t^2. \tag{3.43}$$

Making use of Eqs. (3.41) and (3.42), the x-, y-, and z-components of the above equation are written

$$x = v_0\,\cos\theta\; t, \tag{3.44}$$

$$y = 0, \tag{3.45}$$

$$z = v_0\,\sin\theta\; t - \frac{1}{2}\, g\, t^2, \tag{3.46}$$

respectively. Note that the projectile moves with *constant velocity*, $v_x = dx/dt = v_0\,\cos\theta$, in the x-direction (*i.e.*, horizontally). This is hardly surprising, since there is zero component of the projectile's acceleration along the x-axis. Note, further, that since there is zero component of the projectile's acceleration along the y-axis, and the projectile's initial velocity also has zero component along this axis, the projectile never moves in the y-direction. In other words, the projectile's trajectory is *2-dimensional*, lying entirely within the x-z plane. Note, finally, that the projectile's vertical motion is entirely decoupled from its horizontal motion. In other words, the projectile's vertical motion is identical to that of a second projectile launched vertically upwards, at $t = 0$, with the initial velocity $v_0\,\sin\theta$ (*i.e.*, the initial *vertical* velocity component of the first projectile)—both projectiles will reach the same maximum altitude at the same time, and will subsequently strike the ground simultaneously.

Equations (3.44) and (3.46) can be rearranged to give

$$z = x\,\tan\theta - \frac{1}{2}\frac{g\,x^2}{v_0^2}\,\sec^2\theta. \tag{3.47}$$

As was first pointed out by Galileo, and is illustrated in Fig. 17, this is the equation of a *parabola*. The horizontal range R of the projectile corresponds to its x-coordinate when it strikes the ground (*i.e.*, when $z = 0$). It follows from the above expression (neglecting the trivial result $x = 0$) that

$$R = \frac{2\,v_0^2}{g}\,\sin\theta\,\cos\theta = \frac{v_0^2}{g}\,\sin 2\theta. \tag{3.48}$$

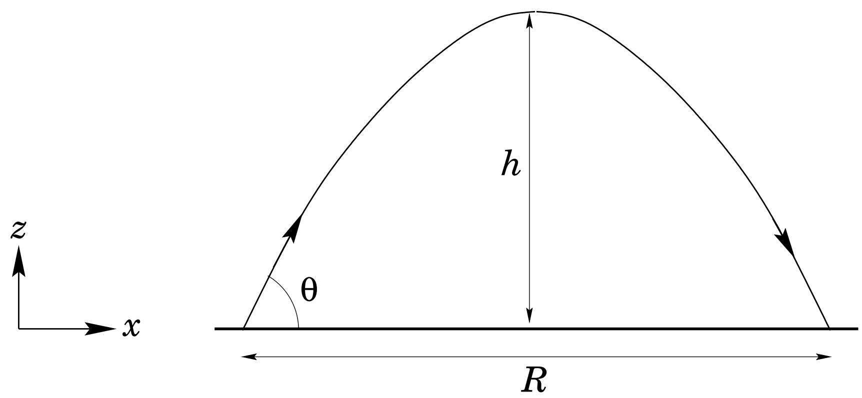


Figure 17: *The parabolic trajectory of a projectile*

Note that the range attains its maximum value,

$$R_{max} = \frac{v_0^2}{g}, \tag{3.49}$$

when $\theta = 45^\circ$. In other words, neglecting air resistance, a projectile travels furthest when it is launched into the air at 45° to the horizontal.

The maximum altitude h of the projectile is attained when $v_z = dz/dt = 0$ (*i.e.*, when the projectile has just stopped rising and is about to start falling). It follows from Eq. (3.46) that the maximum altitude occurs at time $t_0 = v_0 \sin\theta/g$. Hence,

$$h = z(t_0) = \frac{v_0^2}{2\,g} \sin^2\theta. \tag{3.50}$$

Obviously, the largest value of h,

$$h_{max} = \frac{v_0^2}{2\,g}, \tag{3.51}$$

is obtained when the projectile is launched vertically upwards (*i.e.*, $\theta = 90^\circ$).

3.12 Relative velocity

Suppose that, on a windy day, an airplane moves with constant velocity $\mathbf{v}_a$ with respect to the air, and that the air moves with constant velocity $\mathbf{u}$ with respect

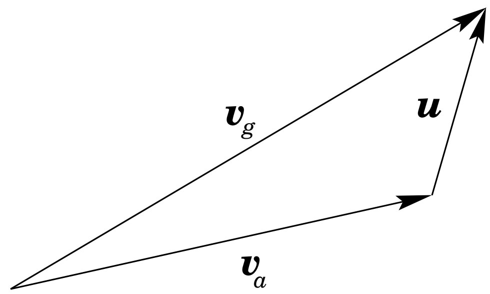


Figure 18: *Relative velocity*

to the ground. What is the vector velocity $\mathbf{v}_g$ of the plane with respect to the ground? In principle, the answer to this question is very simple:

$$\mathbf{v}_g = \mathbf{v}_a + \mathbf{u}. \tag{3.52}$$

In other words, the velocity of the plane with respect to the ground is the vector sum of the plane's velocity relative to the air and the air's velocity relative to the ground. See Fig. 18. Note that, in general, $\mathbf{v}_g$ is parallel to neither $\mathbf{v}_a$ nor $\mathbf{u}$. Let us now consider how we might implement Eq. (3.52) in practice.

As always, our first task is to set up a suitable Cartesian coordinate system. A convenient system for dealing with 2-dimensional motion parallel to the Earth's surface is illustrated in Fig. 19. The x-axis points northward, whereas the y-axis points eastward. In this coordinate system, it is conventional to specify a vector $\mathbf{r}$ in term of its magnitude, r, and its *compass bearing*, ϕ. As illustrated in Fig. 20, a compass bearing is the angle subtended between the direction of a vector and the direction to the North pole: *i.e.*, the x-direction. By convention, compass bearings run from 0° to 360°. Furthermore, the compass bearings of North, East, South, and West are 0°, 90°, 180°, and 270°, respectively.

According to Fig. 20, the components of a general vector $\mathbf{r}$, whose magnitude is r and whose compass bearing is ϕ, are simply

$$\mathbf{r} = (x, y) = (r \cos\phi, r \sin\phi). \tag{3.53}$$

Note that we have suppressed the z-component of $\mathbf{r}$ (which is zero), for ease of notation. Although, strictly speaking, Fig. 20 only justifies the above expression for ϕ in the range 0° to 90°, it turns out that this expression is generally valid: *i.e.*, it is valid for ϕ in the full range 0° to 360°.

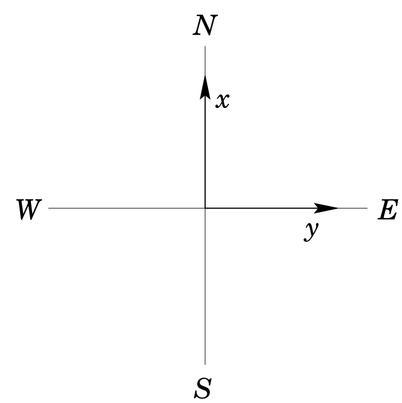


Figure 19: *Coordinates for relative velocity problem*

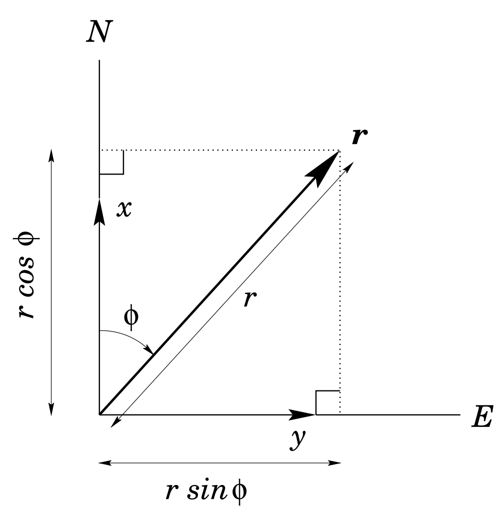


Figure 20: *A compass bearing*

As an illustration, suppose that the plane's velocity relative to the air is 300 km/h, at a compass bearing of 120°, and the air's velocity relative to the ground is 85 km/h, at a compass bearing of 225°. It follows that the components of $\mathbf{v}_a$ and $\mathbf{u}$ (measured in units of km/h) are

$$\mathbf{v}_a = (300\cos 120^\circ,\, 300\sin 120^\circ) = (-1.500\times 10^2,\, 2.598\times 10^2), \quad (3.54)$$
$$\mathbf{u} = (85\cos 225^\circ,\, 85\sin 225^\circ) = (-6.010\times 10^1,\, -6.010\times 10^1). \quad (3.55)$$

According to Eq. (3.52), the components of the plane's velocity $\mathbf{v}_g$ relative to the ground are simply the algebraic sums of the corresponding components of $\mathbf{v}_a$ and $\mathbf{u}$. Hence,

$$\begin{aligned}\mathbf{v}_g &= (-1.500\times 10^2 - 6.010\times 10^1,\, 2.598\times 10^2 - 6.010\times 10^1)\\ &= (-2.101\times 10^2,\, 1.997\times 10^2). \end{aligned}\quad (3.56)$$

Our final task is to reconstruct the magnitude and compass bearing of vector $\mathbf{v}_g$, given its components $(v_{g\,x}, v_{g\,y})$. The magnitude of $\mathbf{v}_g$ follows from Pythagoras' theorem [see Eq. (3.6)]:

$$\begin{aligned}v_g &= \sqrt{(v_{g\,x})^2 + (v_{g\,y})^2}\\ &= \sqrt{(-2.101\times 10^2)^2 + (1.997\times 10^2)^2} = 289.9\,\text{km/h}. \end{aligned}\quad (3.57)$$

In principle, the compass bearing of $\mathbf{v}_g$ is given by the following formula:

$$\phi = \tan^{-1}\left(\frac{v_{g\,y}}{v_{g\,x}}\right). \quad (3.58)$$

This follows because $v_{g\,x} = v_g\cos\phi$ and $v_{g\,y} = v_g\sin\phi$ [see Eq. (3.53)]. Unfortunately, the above expression becomes a little difficult to interpret if $v_{g\,x}$ is negative. An unambiguous pair of expressions for ϕ is given below:

$$\phi = \tan^{-1}\left(\frac{v_{g\,y}}{v_{g\,x}}\right), \quad (3.59)$$

if $v_{g\,x} \geq 0$; or

$$\phi = 180^\circ - \tan^{-1}\left(\frac{v_{g\,y}}{|v_{g\,x}|}\right), \quad (3.60)$$

if $v_{gx} < 0$. These expressions can be derived from simple trigonometry. For the case in hand, Eq. (3.60) is the relevant expression, hence

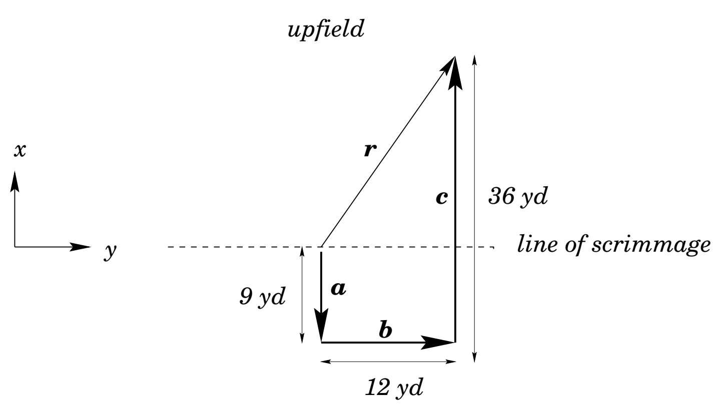
$$\phi = 180^\circ - \tan^{-1}\left(\frac{1.997 \times 10^2}{2.101 \times 10^2}\right) = 136.5^\circ. \tag{3.61}$$

Thus, the plane's velocity relative to the ground is 289.9 km/h at a compass bearing of 136.5°.

Worked example 3.1: Broken play

Question: Major Applewhite receives the snap at the line of scrimmage, takes a seven step drop (*i.e.*, runs backwards 9 yards), but is then flushed out of the pocket by a blitzing linebacker. Major subsequently runs parallel to the line of scrimmage for 12 yards and then gets off a forward pass, 36 yards straight downfield, to Roy Williams, just prior to being creamed by the linebacker. What is the magnitude of the football's resultant displacement (in yards)?

Answer: As illustrated in the diagram, the resultant displacement **r** of the football is the sum of vectors **a**, **b**, and **c**, which correspond to the seven step drop, the run parallel to the line of scrimmage, and the forward pass, respectively. Using



the coordinate system indicated in the diagram, the components of vectors **a**, **b**, and **c** (measured in yards) are

$$\mathbf{a} = (-9, 0),$$

$$\mathbf{b} = (0, 12),$$
$$\mathbf{c} = (36, 0),$$

respectively. Hence the components of **r** are given by

$$\mathbf{r} = (x, y) = (-9 + 0 + 36,\ 0 + 12 + 0) = (27, 12).$$

It follows that the magnitude of the football's resultant displacement is

$$r = \sqrt{x^2 + y^2} = \sqrt{27^2 + 12^2} = 29.55\,\text{yd}.$$

Worked example 3.2: Gallileo's experiment

Question: Legend has it that Gallileo tested out his newly developed theory of projectile motion by throwing weights from the top of the Leaning Tower of Pisa. (No wonder he eventually got into trouble with the authorities!) Suppose that, one day, Gallileo simultaneously threw two equal weights off the tower from a height of 100 m above the ground. Suppose, further, that he dropped the first weight straight down, whereas he threw the second weight horizontally with a velocity of 5 m/s. Which weight struck the ground first? How long, after it was thrown, did it take to do this? Finally, what horizontal distance was traveled by the second weight before it hit the ground? Neglect the effect of air resistance.

Answer: Since both weights start off traveling with the same initial velocities in the vertical direction (*i.e.*, zero), and both accelerate vertically downwards at the same rate, it follows that both weights strike the ground simultaneously. The time of flight of each weight is simply the time taken to fall $h = 100\,\text{m}$, starting from rest, under the influence of gravity. From Eq. (2.17), this time is given by

$$t = \sqrt{\frac{2\,h}{g}} = \sqrt{\frac{2 \times 100}{9.81}} = 4.515\,\text{s}.$$

The horizontal distance R traveled by the second weight is simply the distance traveled by a body moving at a constant velocity $u = 5\,\text{m/s}$ (recall that gravitational acceleration does not affect horizontal motion) during the time taken by the weight to drop 100 m. Thus,

$$R = u\,t = 5 \times 4.515 = 22.58\,\text{m}.$$

Worked example 3.3: Cannon shot

Question: A cannon placed on a 50 m high cliff fires a cannonball over the edge of the cliff at $v = 200\,\mathrm{m/s}$ making an angle of $\theta = 30^\circ$ to the horizontal. How long is the cannonball in the air? Neglect air resistance.

Answer: In order to answer this question we need only consider the cannonball's vertical motion. At $t = 0$ (*i.e.*, the time of firing) the cannonball's height off the ground is $z_0 = 50\,\mathrm{m}$ and its velocity component in the vertical direction is $v_0 = v\,\sin\theta = 200 \times \sin 30^\circ = 100\,\mathrm{m/s}$. Moreover, the cannonball is accelerating vertically downwards at $g = 9.91\,\mathrm{m/s^2}$. The equation of vertical motion of the cannonball is written

$$z = z_0 + v_0\,t - \frac{1}{2}\,g\,t^2,$$

where z is the cannonball's height off the ground at time t. The time of flight of the cannonball corresponds to the time t at which $z = 0$. In other words, the time of flight is the solution of the quadratic equation

$$0 = z_0 + v_0\,t - \frac{1}{2}\,g\,t^2.$$

Hence,

$$t = \frac{v_0 + \sqrt{v_0^2 + 2\,g\,z_0}}{g} = 20.88\,\mathrm{s}.$$

Here, we have neglected the unphysical negative root of our quadratic equation.

Worked example 3.4: Hail Mary pass

Question: The Longhorns are down by 4 points with 5 s left in the fourth quarter. Chris Simms launches a Hail Mary pass into the end-zone, 60 yards away, where B.J. Johnson is waiting to make the catch. Suppose that Chris throws the ball at 55 miles per hour. At what angle to the horizontal must the ball be launched in order for it to hit the receiver? Neglect the effect of air resistance.

Answer: The formula for the horizontal range R of a projectile thrown with initial velocity v_0 at an angle θ to the horizontal is [see Eq. (3.48)]:

$$R = \frac{v_0^2}{g} \sin 2\theta.$$

In this case, $R = 60 \times 3 \times 0.3048 = 54.86\,\text{m}$ and $v_0 = 55 \times 5280 \times 0.3048/3600 = 24.59\,\text{m/s}$. Hence,

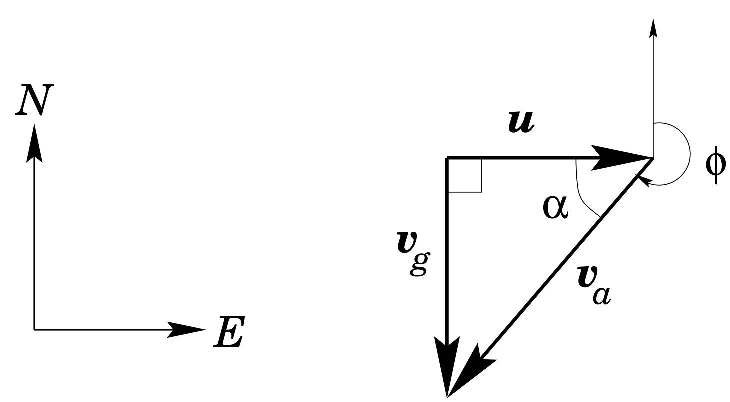
$$\theta = \frac{1}{2} \sin^{-1}\left(\frac{R\,g}{v_0^2}\right) = \frac{1}{2} \sin^{-1}\left(\frac{54.86 \times 9.81}{(24.59)^2}\right) = 31.45^\circ.$$

Thus, the ball must be launched at 31.45° to the horizontal. (Actually, 58.56° would work just as well. Why is this?)

Worked example 3.5: Flight UA 589

Question: United Airlines flight UA 589 from Chicago is 20 miles due North of Austin's Bergstrom airport. Suppose that the plane is flying at 200 mi/h relative to the air. Suppose, further, that there is a wind blowing due East at 60 mi/h. Towards which compass bearing must the plane steer in order to land at the airport?

Answer: The problem in hand is illustrated in the diagram. The plane's veloc-



ity $\mathbf{v}_g$ relative to the ground is the vector sum of its velocity $\mathbf{v}_a$ relative to the air, and the velocity $\mathbf{u}$ of the wind relative to the ground. We know that $\mathbf{u}$ is directed due East, and we require $\mathbf{v}_g$ to be directed due South. We also know that

$|\mathbf{v}_a| = 200$ mi/h and $|\mathbf{u}| = 60$ mi/h. Now, from simple trigonometry,

$$\cos\alpha = \frac{|\mathbf{u}|}{|\mathbf{v}_a|} = \frac{60}{200} = 0.3.$$

Hence,

$$\alpha = 72.54^\circ.$$

However, it is clear from the diagram that the compass bearing ϕ of the plane is given by

$$\phi = 270^\circ - \alpha = 270^\circ - 72.54^\circ = 197.46^\circ.$$

Thus, in order to land at Bergstrom airport the plane must fly towards compass bearing 197.46°.

4 Newton's laws of motion

4.1 Introduction

In his *Principia*, Newton reduced the basic principles of mechanics to three laws:

1. Every body continues in its state of rest, or uniform motion in a straight line, unless compelled to change that state by forces impressed upon it.
2. The change of motion of an object is proportional to the force impressed upon it, and is made in the direction of the straight line in which the force is impressed.
3. To every action there is always opposed an equal reaction; or, the mutual actions of two bodies upon each other are always equal and directed to contrary parts.

These laws are known as Newton's first law of motion, Newton's second law of motion, and Newton's third law of motion, respectively. In this section, we shall examine each of these laws in detail, and then give some simple illustrations of their use.

4.2 Newton's first law of motion

Newton's first law was actually discovered by Galileo and perfected by Descartes (who added the crucial proviso "in a straight line"). This law states that if the motion of a given body is not disturbed by external influences then that body moves with *constant* velocity. In other words, the displacement $\mathbf{r}$ of the body as a function of time t can be written

$$\mathbf{r} = \mathbf{r}_0 + \mathbf{v}\,t, \tag{4.1}$$

where $\mathbf{r}_0$ and $\mathbf{v}$ are *constant* vectors. As illustrated in Fig. 14, the body's trajectory is a *straight-line* which passes through point $\mathbf{r}_0$ at time $t = 0$ and runs parallel to $\mathbf{v}$. In the special case in which $\mathbf{v} = \mathbf{0}$ the body simply remains at rest.

Nowadays, Newton's first law strikes us as almost a statement of the obvious. However, in Galileo's time this was far from being the case. From the time of the ancient Greeks, philosophers—observing that objects set into motion on the Earth's surface eventually come to rest—had concluded that the natural state of motion of objects was that they should remain at rest. Hence, they reasoned, any object which moves does so under the influence of an external influence, or *force*, exerted on it by some other object. It took the genius of Galileo to realize that an object set into motion on the Earth's surface eventually comes to rest under the influence of frictional forces, and that if these forces could somehow be abstracted from the motion then it would continue forever.

4.3 Newton's second law of motion

Newton used the word "motion" to mean what we nowadays call *momentum*. The momentum $\mathbf{p}$ of a body is simply defined as the product of its mass m and its velocity $\mathbf{v}$: *i.e.*,

$$\mathbf{p} = m\,\mathbf{v}. \tag{4.2}$$

Newton's second law of motion is summed up in the equation

$$\frac{d\mathbf{p}}{dt} = \mathbf{f}, \tag{4.3}$$

where the vector $\mathbf{f}$ represents the net influence, or force, exerted on the object, whose motion is under investigation, by other objects. For the case of a object with *constant* mass, the above law reduces to its more conventional form

$$\mathbf{f} = m\,\mathbf{a}. \tag{4.4}$$

In other words, the net force exerted on a given object by other objects equals the product of that object's mass and its acceleration. Of course, this law is entirely devoid of content unless we have some independent means of quantifying the forces exerted between different objects.

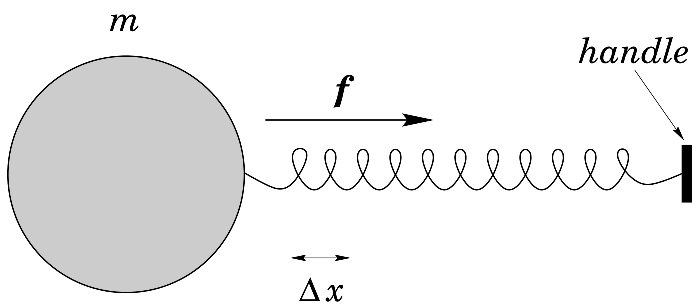


Figure 21: *Hooke's law*

4.4 Hooke's law

One method of quantifying the force exerted on an object is via *Hooke's law*. This law—discovered by the English scientist Robert Hooke in 1660—states that the force f exerted by a coiled spring is directly proportional to its *extension* Δx. The extension of the spring is the difference between its actual length and its natural length (*i.e.*, its length when it is exerting no force). The force acts parallel to the axis of the spring. Obviously, Hooke's law only holds if the extension of the spring is sufficiently small. If the extension becomes too large then the spring deforms permanently, or even breaks. Such behaviour lies beyond the scope of Hooke's law.

Figure 21 illustrates how we might use Hooke's law to quantify the force we exert on a body of mass m when we pull on the handle of a spring attached to it. The magnitude f of the force is proportional to the extension of the spring: twice the extension means twice the force. As shown, the direction of the force is towards the spring, parallel to its axis (assuming that the extension is positive). The magnitude of the force can be quantified in terms of the critical extension required to impart a unit acceleration (*i.e.*, $1\,m/s^2$) to a body of unit mass (*i.e.*, $1\,kg$). According to Eq. (4.4), the force corresponding to this extension is 1 *newton*. Here, a newton (symbol N) is equivalent to a kilogram-meter per second-squared, and is the mks unit of force. Thus, if the critical extension corresponds to a force of $1\,N$ then half the critical extension corresponds to a force of $0.5\,N$, and so on. In this manner, we can quantify both the direction and magnitude of the force we exert, by means of a spring, on a given body.

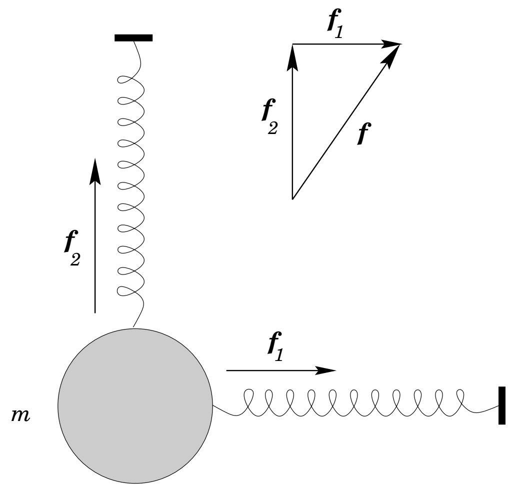


Figure 22: *Addition of forces*

Suppose that we apply two forces, $\mathbf{f}_1$ and $\mathbf{f}_2$ (say), acting in different directions, to a body of mass m by means of two springs. As illustrated in Fig. 22, the body accelerates as if it were subject to a single force $\mathbf{f}$ which is the *vector sum* of the individual forces $\mathbf{f}_1$ and $\mathbf{f}_2$. It follows that the force $\mathbf{f}$ appearing in Newton's second law of motion, Eq. (4.4), is the *resultant* of all the external forces to which the body whose motion is under investigation is subject.

Suppose that the resultant of all the forces acting on a given body is *zero*. In other words, suppose that the forces acting on the body exactly balance one another. According to Newton's second law of motion, Eq. (4.4), the body does not accelerate: *i.e.*, it either remains at rest or moves with uniform velocity in a straight line. It follows that Newton's first law of motion applies not only to bodies which have no forces acting upon them but also to bodies acted upon by exactly balanced forces.

4.5 Newton's third law of motion

Suppose, for the sake of argument, that there are only two bodies in the Universe. Let us label these bodies a and b. Suppose that body b exerts a force $\mathbf{f}_{ab}$ on body a. According to to Newton's third law of motion, body a must exert an *equal and opposite* force $\mathbf{f}_{ba} = -\mathbf{f}_{ab}$ on body b. See Fig. 22. Thus, if we label $\mathbf{f}_{ab}$ the "action"

Figure 23: *Newton's third law*

then, in Newton's language, $\mathbf{f}_{ba}$ is the equal and opposed "reaction".

Suppose, now, that there are many objects in the Universe (as is, indeed, the case). According to Newton's third law, if object j exerts a force $\mathbf{f}_{ij}$ on object i then object i must exert an equal and opposite force $\mathbf{f}_{ji} = -\mathbf{f}_{ij}$ on object j. It follows that all of the forces acting in the Universe can ultimately be grouped into equal and opposite action-reaction pairs. Note, incidentally, that an action and its associated reaction always act on *different* bodies.

Why do we need Newton's third law? Actually, it is almost a matter of common sense. Suppose that bodies a and b constitute an *isolated* system. If $\mathbf{f}_{ba} \neq -\mathbf{f}_{ab}$ then this system exerts a *non-zero net force* $\mathbf{f} = \mathbf{f}_{ab} + \mathbf{f}_{ba}$ on itself, without the aid of any external agency. It will, therefore, accelerate forever under its own steam. We know, from experience, that this sort of behaviour does not occur in real life. For instance, I cannot grab hold of my shoelaces and, thereby, pick myself up off the ground. In other words, I cannot self-generate a force which will spontaneously lift me into the air: I need to exert forces on other objects around me in order to achieve this. Thus, Newton's third law essentially acts as a guarantee against the absurdity of self-generated forces.

4.6 Mass and weight

The terms *mass* and *weight* are often confused with one another. However, in physics their meanings are quite distinct.

A body's mass is a measure of its *inertia*: *i.e.*, its reluctance to deviate from uniform straight-line motion under the influence of external forces. According to Newton's second law, Eq. (4.4), if two objects of differing masses are acted upon

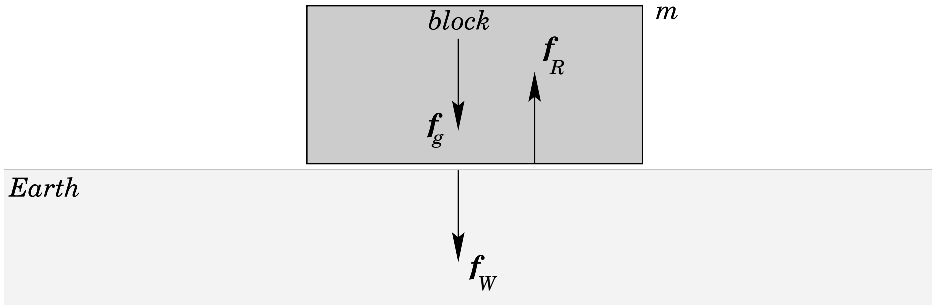


Figure 24: *Weight*

by forces of the same magnitude then the resulting acceleration of the larger mass is less than that of the smaller mass. In other words, it is more difficult to force the larger mass to deviate from its preferred state of uniform motion in a straight line. Incidentally, the mass of a body is an intrinsic property of that body, and, therefore, does not change if the body is moved to a different place.

Imagine a block of granite resting on the surface of the Earth. See Fig. 24. The block experiences a downward force $\mathbf{f}_g$ due to the gravitational attraction of the Earth. This force is of magnitude $m\,g$, where m is the mass of the block and g is the acceleration due to gravity at the surface of the Earth. The block transmits this force to the ground below it, which is supporting it, and, thereby, preventing it from accelerating downwards. In other words, the block exerts a downward force $\mathbf{f}_W$, of magnitude $m\,g$, on the ground immediately beneath it. We usually refer to this force (or the magnitude of this force) as the *weight* of the block. According to Newton's third law, the ground below the block exerts an upward reaction force $\mathbf{f}_R$ on the block. This force is also of magnitude $m\,g$. Thus, the net force acting on the block is $\mathbf{f}_g + \mathbf{f}_R = \mathbf{0}$, which accounts for the fact that the block remains stationary.

Where, you might ask, is the equal and opposite reaction to the force of gravitational attraction $\mathbf{f}_g$ exerted by the Earth on the block of granite? It turns out that this reaction is exerted at the centre of the Earth. In other words, the Earth attracts the block of granite, and the block of granite attracts the Earth by an equal amount. However, since the Earth is far more massive than the block, the force exerted by the granite block at the centre of the Earth has no observable consequence.

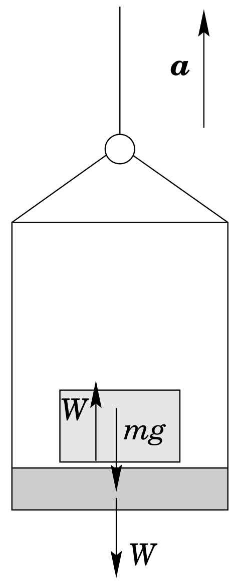


Figure 25: *Weight in an elevator*

So far, we have established that the weight W of a body is the magnitude of the downward force it exerts on any object which supports it. Thus, $W = m\,g$, where m is the mass of the body and g is the local acceleration due to gravity. Since weight is a force, it is measured in newtons. A body's weight is location dependent, and is not, therefore, an intrinsic property of that body. For instance, a body weighing 10 N on the surface of the Earth will only weigh about 3.8 N on the surface of Mars, due to the weaker surface gravity of Mars relative to the Earth.

Consider a block of mass m resting on the floor of an elevator, as shown in Fig. 25. Suppose that the elevator is accelerating upwards with acceleration a. How does this acceleration affect the weight of the block? Of course, the block experiences a downward force $m\,g$ due to gravity. Let W be the weight of the block: by definition, this is the size of the downward force exerted by the block on the floor of the elevator. From Newton's third law, the floor of the elevator exerts an upward reaction force of magnitude W on the block. Let us apply Newton's second law, Eq. (4.4), to the motion of the block. The mass of the block is m, and its upward acceleration is a. Furthermore, the block is subject to two forces: a downward force $m\,g$ due to gravity, and an upward reaction force W.

Hence,

$$W - m\,g = m\,a. \tag{4.5}$$

This equation can be rearranged to give

$$W = m\,(g + a). \tag{4.6}$$

Clearly, the upward acceleration of the elevator has the effect of increasing the weight W of the block: for instance, if the elevator accelerates upwards at $g = 9.81\,\mathrm{m/s^2}$ then the weight of the block is doubled. Conversely, if the elevator accelerates downward (*i.e.*, if a becomes negative) then the weight of the block is reduced: for instance, if the elevator accelerates downward at $g/2$ then the weight of the block is halved. Incidentally, these weight changes could easily be measured by placing some scales between the block and the floor of the elevator.

Suppose that the downward acceleration of the elevator matches the acceleration due to gravity: *i.e.*, $a = -g$. In this case, $W = 0$. In other words, the block becomes weightless! This is the principle behind the so-called "Vomit Comet" used by NASA's Johnson Space Centre to train prospective astronauts in the effects of weightlessness. The "Vomit Comet" is actually a KC-135 (a predecessor of the Boeing 707 which is typically used for refueling military aircraft). The plane typically ascends to 30,000 ft and then accelerates downwards at g (*i.e.*, drops like a stone) for about 20 s, allowing its passengers to feel the effects of weightlessness during this period. All of the weightless scenes in the film *Apollo 11* were shot in this manner.

Suppose, finally, that the downward acceleration of the elevator exceeds the acceleration due to gravity: *i.e.*, $a < -g$. In this case, the block acquires a negative weight! What actually happens is that the block flies off the floor of the elevator and slams into the ceiling: when things have settled down, the block exerts an upward force (negative weight) $|W|$ on the ceiling of the elevator.

4.7 Strings, pulleys, and inclines

Consider a block of mass m which is suspended from a fixed beam by means of a string, as shown in Fig. 26. The string is assumed to be light (*i.e.*, its mass

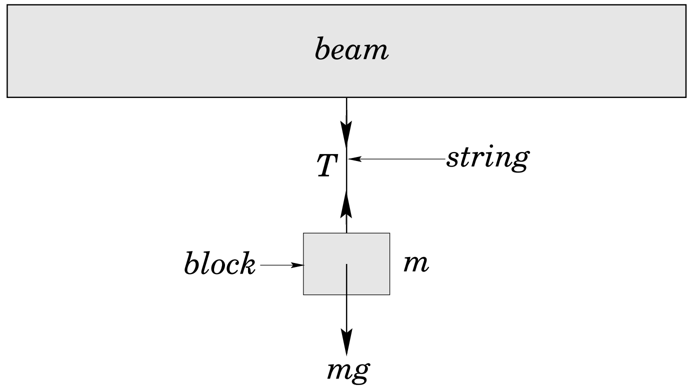


Figure 26: *Block suspended by a string*

is negligible compared to that of the block) and inextensible (*i.e.*, its length increases by a negligible amount because of the weight of the block). The string is clearly being stretched, since it is being pulled at both ends by the block and the beam. Furthermore, the string must be being pulled by oppositely directed forces of the same magnitude, otherwise it would accelerate greatly (given that it has negligible inertia). By Newton's third law, the string exerts oppositely directed forces of equal magnitude, T (say), on both the block and the beam. These forces act so as to oppose the stretching of the string: *i.e.*, the beam experiences a downward force of magnitude T, whereas the block experiences an upward force of magnitude T. Here, T is termed the *tension* of the string. Since T is a force, it is measured in newtons. Note that, unlike a coiled spring, a string can never possess a negative tension, since this would imply that the string is trying to push its supports apart, rather than pull them together.

Let us apply Newton's second law to the block. The mass of the block is m, and its acceleration is zero, since the block is assumed to be in equilibrium. The block is subject to two forces, a downward force $m\,g$ due to gravity, and an upward force T due to the tension of the string. It follows that

$$T - m\,g = 0. \tag{4.7}$$

In other words, in equilibrium, the tension T of the string equals the weight $m\,g$ of the block.

Figure 27 shows a slightly more complicated example in which a block of mass m is suspended by three strings. The question is what are the tensions, T, T_1, and

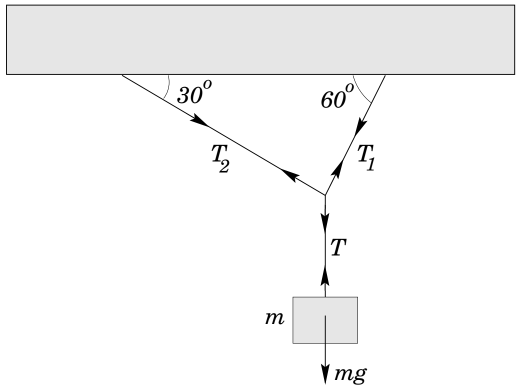


Figure 27: *Block suspended by three strings*

T_2, in these strings, assuming that the block is in equilibrium? Using analogous arguments to the previous case, we can easily demonstrate that the tension T in the lowermost string is $m\,g$. The tensions in the two uppermost strings are obtained by applying Newton's second law of motion to the knot where all three strings meet. See Fig. 28.

There are three forces acting on the knot: the downward force T due to the tension in the lower string, and the forces T_1 and T_2 due to the tensions in the upper strings. The latter two forces act along their respective strings, as indicate in the diagram. Since the knot is in equilibrium, the vector sum of all the forces acting on it must be zero.

Consider the horizontal components of the forces acting on the knot. Let components acting to the right be positive, and *vice versa*. The horizontal component of tension T is zero, since this tension acts straight down. The horizontal component of tension T_1 is $T_1 \cos 60^\circ = T_1/2$, since this force subtends an angle of 60° with respect to the horizontal (see Fig. 16). Likewise, the horizontal component of tension T_2 is $-T_2 \cos 30^\circ = -\sqrt{3}\,T_2/2$. Since the knot does not accelerate in the horizontal direction, we can equate the sum of these components to zero:

$$\frac{T_1}{2} - \frac{\sqrt{3}\,T_2}{2} = 0. \tag{4.8}$$

Consider the vertical components of the forces acting on the knot. Let components acting upward be positive, and *vice versa*. The vertical component of

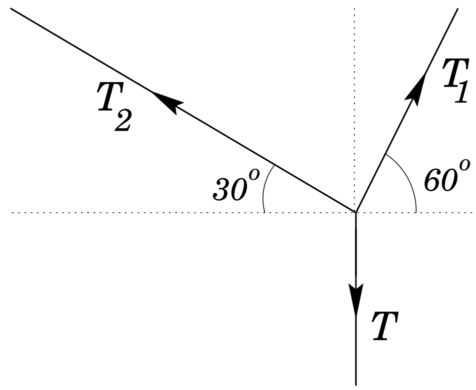


Figure 28: *Detail of Fig. 27*

tension T is $-T = -m\,g$, since this tension acts straight down. The vertical component of tension T_1 is $T_1\,\sin 60^\circ = \sqrt{3}\,T_1/2$, since this force subtends an angle of 60° with respect to the horizontal (see Fig. 16). Likewise, the vertical component of tension T_2 is $T_2\,\sin 30^\circ = T_2/2$. Since the knot does not accelerate in the vertical direction, we can equate the sum of these components to zero:

$$-\,m\,g + \frac{\sqrt{3}\,T_1}{2} + \frac{T_2}{2} = 0. \tag{4.9}$$

Finally, Eqs. (4.8) and (4.9) yield

$$T_1 = \frac{\sqrt{3}\,m\,g}{2}, \tag{4.10}$$

$$T_2 = \frac{m\,g}{2}. \tag{4.11}$$

Consider a block of mass m sliding down a smooth frictionless incline which subtends an angle θ to the horizontal, as shown in Fig 29. The weight $m\,g$ of the block is directed vertically downwards. However, this force can be resolved into components $m\,g\,\cos\theta$, acting perpendicular (or normal) to the incline, and $m\,g\,\sin\theta$, acting parallel to the incline. Note that the reaction of the incline to the weight of the block acts *normal* to the incline, and only matches the *normal component* of the weight: *i.e.*, it is of magnitude $m\,g\,\cos\theta$. This is a general result: the reaction of any unyielding surface is always locally normal to that surface, directed outwards (away from the surface), and matches the normal

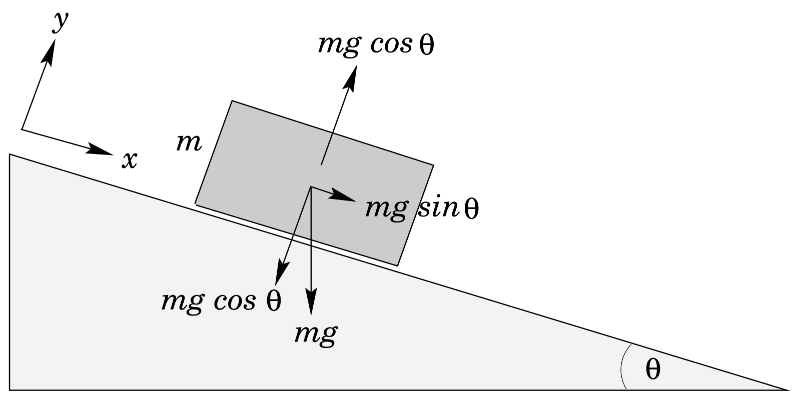


Figure 29: *Block sliding down an incline*

component of any inward force applied to the surface. The block is clearly in equilibrium in the direction normal to the incline, since the normal component of the block's weight is balanced by the reaction of the incline. However, the block is subject to the unbalanced force $m\,g\,\sin\theta$ in the direction parallel to the incline, and, therefore, accelerates down the slope. Applying Newton's second law to this problem (with the coordinates shown in the figure), we obtain

$$m\,\frac{d^2x}{dt^2} = m\,g\,\sin\theta, \tag{4.12}$$

which can be solved to give

$$x = x_0 + v_0\,t + \frac{1}{2}\,g\,\sin\theta\,t^2. \tag{4.13}$$

In other words, the block accelerates down the slope with acceleration $g\,\sin\theta$. Note that this acceleration is *less* than the full acceleration due to gravity, g. In fact, if the incline is fairly gentle (*i.e.*, if θ is small) then the acceleration of the block can be made *much less* than g. This was the technique used by Galileo in his pioneering studies of motion under gravity—by diluting the acceleration due to gravity, using inclined planes, he was able to obtain motion sufficiently slow for him to make accurate measurements using the crude time-keeping devices available in the 17th Century.

Consider two masses, m_1 and m_2, connected by a light inextensible string. Suppose that the first mass slides over a smooth, frictionless, horizontal table, whilst the second is suspended over the edge of the table by means of a light frictionless pulley. See Fig. 30. Since the pulley is light, we can neglect its rotational inertia in our analysis. Moreover, no force is required to turn a frictionless pulley, so we can assume that the tension T of the string is the same on either side of the pulley. Let us apply Newton's second law of motion to each mass in turn. The first mass is subject to a downward force $m_1\, g$, due to gravity. However, this force is completely canceled out by the upward reaction force due to the table. The mass m_1 is also subject to a horizontal force T, due to the tension in the string, which causes it to move *rightwards* with acceleration

$$a = \frac{T}{m_1}. \tag{4.14}$$

The second mass is subject to a downward force $m_2\, g$, due to gravity, plus an upward force T due to the tension in the string. These forces cause the mass to move *downwards* with acceleration

$$a = g - \frac{T}{m_2}. \tag{4.15}$$

Now, the rightward acceleration of the first mass must match the downward acceleration of the second, since the string which connects them is inextensible. Thus, equating the previous two expressions, we obtain

$$T = \frac{m_1\, m_2}{m_1 + m_2}\, g, \tag{4.16}$$

$$a = \frac{m_2}{m_1 + m_2}\, g. \tag{4.17}$$

Note that the acceleration of the two coupled masses is *less* than the full acceleration due to gravity, g, since the first mass contributes to the inertia of the system, but does not contribute to the downward gravitational force which sets the system in motion.

Consider two masses, m_1 and m_2, connected by a light inextensible string which is suspended from a light frictionless pulley, as shown in Fig. 31. Let us again apply Newton's second law to each mass in turn. Without being given the values of m_1 and m_2, we cannot determine beforehand which mass is going to move upwards. Let us *assume* that mass m_1 is going to move upwards: if we are

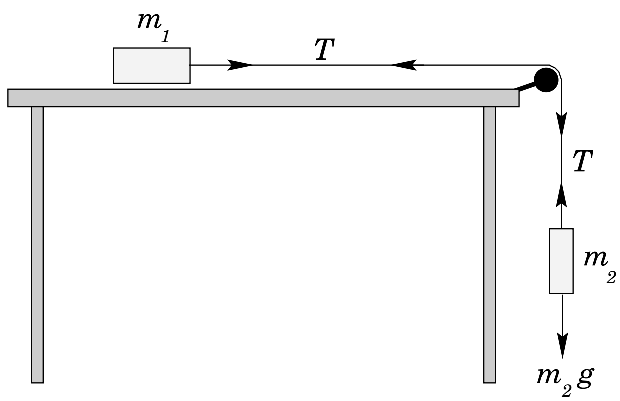


Figure 30: *Block sliding over a smooth table, pulled by a second block*

wrong in this assumption then we will simply obtain a negative acceleration for this mass. The first mass is subject to an upward force T, due to the tension in the string, and a downward force $m_1\, g$, due to gravity. These forces cause the mass to move *upwards* with acceleration

$$a = \frac{T}{m_1} - g. \tag{4.18}$$

The second mass is subject to a downward force $m_2\, g$, due to gravity, and an upward force T, due to the tension in the string. These forces cause the mass to move *downward* with acceleration

$$a = g - \frac{T}{m_2}. \tag{4.19}$$

Now, the upward acceleration of the first mass must match the downward acceleration of the second, since they are connected by an inextensible string. Hence, equating the previous two expressions, we obtain

$$T = \frac{2\, m_1\, m_2}{m_1 + m_2}\, g, \tag{4.20}$$

$$a = \frac{m_2 - m_1}{m_1 + m_2}\, g. \tag{4.21}$$

As expected, the first mass accelerates upward (*i.e.*, $a > 0$) if $m_2 > m_1$, and *vice versa*. Note that the acceleration of the system is *less* than the full acceleration due to gravity, g, since both masses contribute to the inertia of the system, but

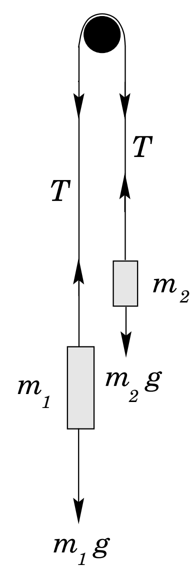


Figure 31: *An Atwood machine*

their weights partially cancel one another out. In particular, if the two masses are almost equal then the acceleration of the system becomes very much less than g.

Incidentally, the device pictured in Fig. 31 is called an *Atwood machine*, after the eighteenth Century English scientist George Atwood, who used it to "slow down" free-fall sufficiently to make accurate observations of this phenomena using the primitive time-keeping devices available in his day.

4.8 Friction

When a body slides over a rough surface a frictional force generally develops which acts to impede the motion. Friction, when viewed at the microscopic level, is actually a very complicated phenomenon. Nevertheless, physicists and engineers have managed to develop a relatively simple empirical law of force which allows the effects of friction to be incorporated into their calculations. This law of force was first proposed by Leonardo da Vinci (1452–1519), and later extended by Charles Augustin de Coulomb (1736–1806) (who is more famous for discovering the law of electrostatic attraction). The frictional force exerted on a body sliding over a rough surface is proportional to the *normal reaction* R_n at that surface, the constant of proportionality depending on the nature of the surface. In other words,

$$f = \mu R_n, \tag{4.22}$$

where μ is termed the *coefficient of (dynamical) friction*. For ordinary surfaces, μ is generally of order unity.

Consider a block of mass m being dragged over a horizontal surface, whose coefficient of friction is μ, by a horizontal force F. See Fig. 32. The weight $W = m\,g$ of the block acts vertically downwards, giving rise to a reaction $R = m\,g$ acting vertically upwards. The magnitude of the frictional force f, which impedes the motion of the block, is simply μ times the normal reaction $R = m\,g$. Hence, $f = \mu\,m\,g$. The acceleration of the block is, therefore,

$$a = \frac{F - f}{m} = \frac{F}{m} - \mu\,g, \tag{4.23}$$

assuming that $F > f$. What happens if $F < f$: *i.e.*, if the applied force F is less than the frictional force f? In this case, common sense suggests that the block simply remains at rest (it certainly does not accelerate backwards!). Hence, $f = \mu\,m\,g$ is actually the *maximum* force which friction can generate in order to impede the motion of the block. If the applied force F is less than this maximum value then the applied force is canceled out by an equal and opposite frictional force, and the block remains stationary. Only if the applied force exceeds the maximum frictional force does the block start to move.

Consider a block of mass m sliding down a rough incline (coefficient of friction μ) which subtends an angle θ to the horizontal, as shown in Fig 33. The weight $m\,g$ of the block can be resolved into components $m\,g\,\cos\theta$, acting normal to the incline, and $m\,g\,\sin\theta$, acting parallel to the incline. The reaction of the incline to the weight of the block acts *normally* outwards from the incline, and is of magnitude $m\,g\,\cos\theta$. Parallel to the incline, the block is subject to the downward gravitational force $m\,g\,\sin\theta$, and the upward frictional force f (which acts to prevent the block sliding down the incline). In order for the block to move, the magnitude of the former force must exceed the maximum value of the latter,

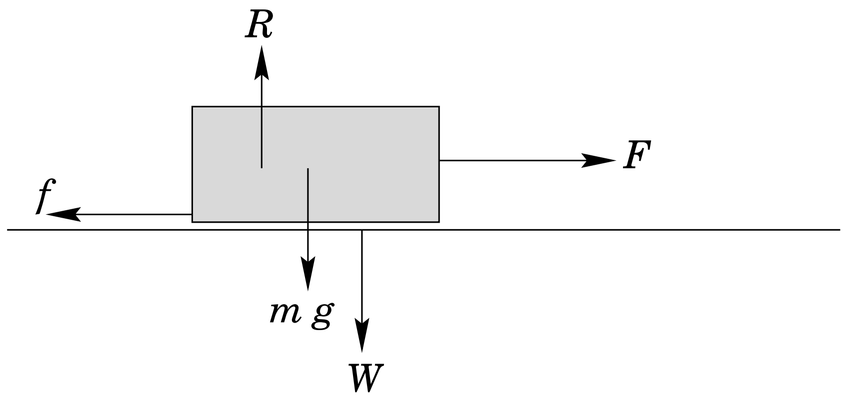


Figure 32: *Friction*

which is μ time the magnitude of the normal reaction, or $\mu\, m\, g\, \cos\theta$. Hence, the condition for the weight of the block to overcome friction, and, thus, to cause the block to slide down the incline, is

$$m\, g\, \sin\theta > \mu\, m\, g\, \cos\theta, \tag{4.24}$$

or

$$\tan\theta > \mu. \tag{4.25}$$

In other words, if the slope of the incline exceeds a certain critical value, which depends on μ, then the block will start to slide. Incidentally, the above formula suggests a fairly simple way of determining the coefficient of friction for a given object sliding over a particular surface. Simply tilt the surface gradually until the object just starts to move: the coefficient of friction is simply the tangent of the critical tilt angle (measured with respect to the horizontal).

Up to now, we have implicitly suggested that the coefficient of friction between an object and a surface is the same whether the object remains stationary or slides over the surface. In fact, this is generally not the case. Usually, the coefficient of friction when the object is stationary is slightly *larger* than the coefficient when the object is sliding. We call the former coefficient the *coefficient of static friction*, μ_s, whereas the latter coefficient is usually termed the *coefficient of kinetic (or dynamical) friction*, μ_k. The fact that $\mu_s > \mu_k$ simply implies that objects have a tendency to "stick" to rough surfaces when placed upon them. The force required to unstick a given object, and, thereby, set it in motion, is μ_s times the normal

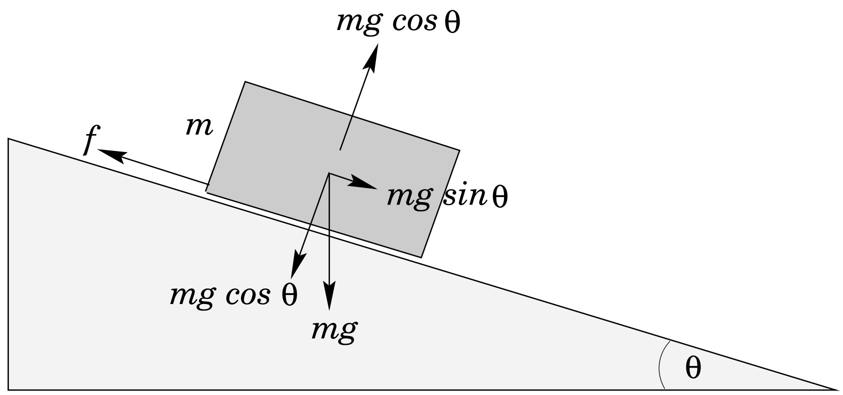


Figure 33: *Block sliding down a rough slope*

reaction at the surface. Once the object has been set in motion, the frictional force acting to impede this motion falls somewhat to μ_k times the normal reaction.

4.9 Frames of reference

As discussed in Sect. 1, the laws of physics are assumed to possess *objective reality*. In other words, it is assumed that two independent observers, studying the same physical phenomenon, would eventually formulate *identical* laws of physics in order to account for their observations. Now, two completely independent observers are likely to choose different systems of units with which to quantify physical measurements. However, as we have seen in Sect. 1, the dimensional consistency of valid laws of physics renders them *invariant* under transformation from one system of units to another. Independent observers are also likely to choose different coordinate systems. For instance, the origins of their separate coordinate systems might differ, as well as the orientation of the various coordinate axes. Are the laws of physics also *invariant* under transformation between coordinate systems possessing different origins, or a different orientation of the various coordinate axes?

Consider the vector equation

$$\mathbf{r} = \mathbf{r}_1 + \mathbf{r}_2, \tag{4.26}$$

which is represented diagrammatically in Fig. 12. Suppose that we shift the origin of our coordinate system, or rotate the coordinate axes. Clearly, in general, the *components* of vectors $\mathbf{r}$, $\mathbf{r}_1$, and $\mathbf{r}_2$ are going to be modified by this change in our coordinate scheme. However, Fig. 12 still remains valid. Hence, we conclude that the vector equation (4.26) also remains valid. In other words, although the individual components of vectors $\mathbf{r}$, $\mathbf{r}_1$, and $\mathbf{r}_2$ are modified by the change in coordinate scheme, the interrelation between these components expressed in Eq. (4.26) remains invariant. This observation suggests that the independence of the laws of physics from the arbitrary choice of the location of the underlying coordinate system's origin, or the equally arbitrary choice of the orientation of the various coordinate axes, can be made manifest by simply writing these laws as interrelations between vectors. In particular, Newton's second law of motion,

$$\mathbf{f} = m\,\mathbf{a}, \tag{4.27}$$

is clearly invariant under shifts in the origin of our coordinate system, or changes in the orientation of the various coordinate axes. Note that the quantity m (*i.e.*, the mass of the body whose motion is under investigation), appearing in the above equation, is invariant under any changes in the coordinate system, since measurements of mass are completely independent of measurements of distance. We refer to such a quantity as a *scalar* (this is an improved definition). We conclude that valid laws of physics must consist of combinations of scalars and vectors, otherwise they would retain an unphysical dependence on the details of the chosen coordinate system.

Up to now, we have implicitly assumed that all of our observers are *stationary* (*i.e.*, they are all standing still on the surface of the Earth). Let us, now, relax this assumption. Consider two observers, O and O', whose coordinate systems coincide momentarily at $t = 0$. Suppose that observer O is stationary (on the surface of the Earth), whereas observer O' moves (with respect to observer O) with *uniform* velocity $\mathbf{v}_0$. As illustrated in Fig. 34, if $\mathbf{r}$ represents the displacement of some body P in the stationary observer's frame of reference, at time t, then the corresponding displacement in the moving observer's frame of reference is simply

$$\mathbf{r}' = \mathbf{r} - \mathbf{v}_0\,t. \tag{4.28}$$

The velocity of body P in the stationary observer's frame of reference is defined

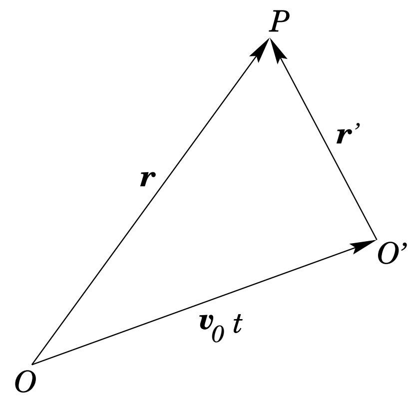


Figure 34: *A moving observer*

as

$$\mathbf{v} = \frac{d\mathbf{r}}{dt}. \tag{4.29}$$

Hence, the corresponding velocity in the moving observer's frame of reference takes the form

$$\mathbf{v}' = \frac{d\mathbf{r}'}{dt} = \mathbf{v} - \mathbf{v}_0. \tag{4.30}$$

Finally, the acceleration of body P in stationary observer's frame of reference is defined as

$$\mathbf{a} = \frac{d\mathbf{v}}{dt}, \tag{4.31}$$

whereas the corresponding acceleration in the moving observer's frame of reference takes the form

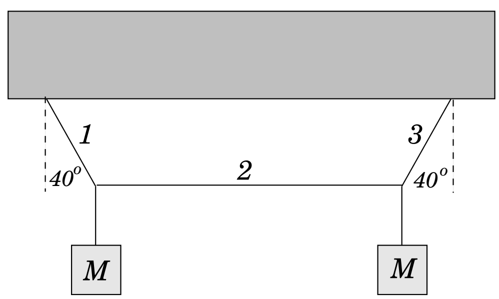
$$\mathbf{a}' = \frac{d\mathbf{v}'}{dt} = \mathbf{a}. \tag{4.32}$$

Hence, the acceleration of body P is *identical* in both frames of reference.

It is clear that if observer O concludes that body P is moving with constant velocity, and, therefore, subject to zero net force, then observer O' will agree with this conclusion. Furthermore, if observer O concludes that body P is accelerating, and, therefore, subject to a force $\mathbf{a}/m$, then observer O' will remain in agreement. It follows that Newton's laws of motion are equally valid in the frames of reference of the moving and the stationary observer. Such frames are termed *inertial frames of reference*. There are infinitely many inertial frames of reference—within which Newton's laws of motion are equally valid—all moving with *constant* velocity with respect to one another. Consequently, there is no universal standard of rest in physics. Observer O might claim to be at rest compared to observer O′, and *vice versa*: however, both points of view are equally valid. Moreover, there is absolutely no physical experiment which observer O could perform in order to demonstrate that he/she is at rest whilst observer O′ is moving. This, in essence, is the principle of *special relativity*, first formulated by Albert Einstein in 1905.

Worked example 4.1: In equilibrium

Question: Consider the diagram. If the system is in equilibrium, and the tension in string 2 is 50 N, determine the mass M.



Answer: It follows from symmetry that the tensions in strings 1 and 3 are equal. Let T_1 be the tension in string 1, and T_2 the tension in string 2. Consider the equilibrium of the knot above the leftmost mass. As shown below, this knot is subject to three forces: the downward force $T_4 = M\,g$ due to the tension in the string which directly supports the leftmost mass, the rightward force T_2 due to the tension in string 2, and the upward and leftward force T_1 due to the tension in string 1. The resultant of all these forces must be zero, otherwise the system would not be in equilibrium. Resolving in the horizontal direction (with rightward forces positive), we obtain

$$T_2 - T_1\,\sin 40^\circ = 0.$$

Likewise, resolving in the vertical direction (with upward forces positive) yields

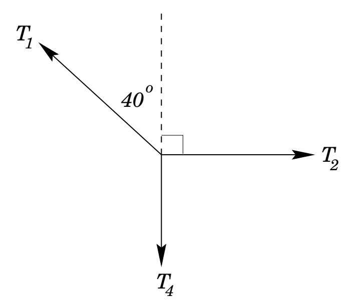
$$T_1\,\cos 40^\circ - T_4 = 0.$$

Combining the above two expressions, making use of the fact that $T_4 = M\,g$, gives

$$M = \frac{T_2}{g\,\tan 40^\circ}.$$

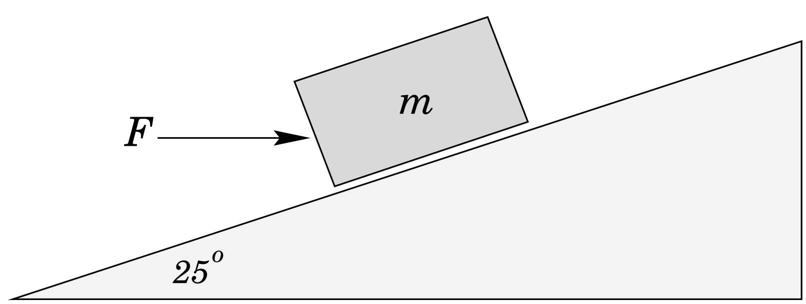
Finally, since $T_2 = 50\,\mathrm{N}$ and $g = 9.81\,\mathrm{m/s^2}$, we obtain

$$M = \frac{50}{9.81 \times 0.8391} = 6.074\,\mathrm{kg}.$$



Worked example 4.2: Block accelerating up a slope

Question: Consider the diagram. Suppose that the block, mass $m = 5\,\mathrm{kg}$, is subject to a horizontal force $F = 27\,\mathrm{N}$. What is the acceleration of the block up the (frictionless) slope?



Answer: Only that component of the applied force which is parallel to the incline has any influence on the block's motion: the normal component of the applied force is canceled out by the normal reaction of the incline. The component of the applied force acting up the incline is $F\,\cos 25^\circ$. Likewise, the component of the block's weight acting down the incline is $m\,g\,\sin 25^\circ$. Hence, using Newton's second law to determine the acceleration a of the block up the incline, we obtain

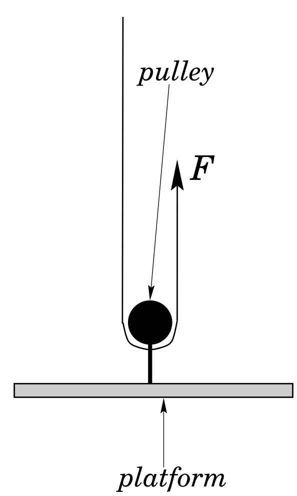
$$a = \frac{F\,\cos 25^\circ - m\,g\,\sin 25^\circ}{m}.$$

Since $m = 5\,\mathrm{kg}$ and $F = 27\,\mathrm{N}$, we have

$$a = \frac{27 \times 0.9063 - 5 \times 9.81 \times 0.4226}{5} = 0.7483\,\mathrm{m/s^2}.$$

Worked example 4.3: Raising a platform

Question: Consider the diagram. The platform and the attached frictionless pulley weigh a total of 34 N. With what force F must the (light) rope be pulled in order to lift the platform at $3.2\,\mathrm{m/s^2}$?



Answer: Let W be the weight of the platform, $m = W/g$ the mass of the platform, and T the tension in the rope. From Newton's third law, it is clear that $T = F$. Let us apply Newton's second law to the upward motion of the platform. The platform is subject to two vertical forces: a downward force W due to its weight, and an upward force $2\,T$ due to the tension in the rope (the force is $2\,T$, rather than T, because both the leftmost and rightmost sections of the rope, emerging from the pulley, are in tension and exerting an upward force on the pulley). Thus, the upward acceleration a of the platform is

$$a = \frac{2T - W}{m}.$$

Since $T = F$ and $m = W/g$, we obtain

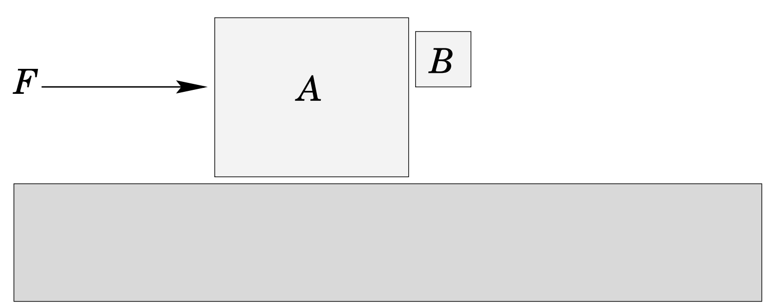
$$F = \frac{W\,(a/g + 1)}{2}.$$

Finally, given that $W = 34\,\mathrm{N}$ and $a = 3.2\,\mathrm{m/s^2}$, we have

$$F = \frac{34\,(3.2/9.81 + 1)}{2} = 22.55\,\mathrm{N}.$$

Worked example 4.4: Suspended block

Question: Consider the diagram. The mass of block A is $75\,\mathrm{kg}$ and the mass of block B is $15\,\mathrm{kg}$. The coefficient of static friction between the two blocks is $\mu = 0.45$. The horizontal surface is frictionless. What minimum force F must be exerted on block A in order to prevent block B from falling?



Answer: Suppose that block A exerts a rightward force R on block B. By Newton's third law, block B exerts an equal and opposite force on block A. Applying Newton's second law of motion to the rightward acceleration a of block B, we obtain

$$a = \frac{R}{m_B},$$

where m_B is the mass of block B. The normal reaction at the interface between the two blocks is R. Hence, the maximum frictional force that block A can exert on block B is $\mu\,R$. In order to prevent block B from falling, this maximum frictional force (which acts upwards) must exceed the downward acting weight, $m_B\, g$, of the block. Hence, we require

$$\mu\, R > m_B\, g,$$

or

$$a > \frac{g}{\mu}.$$

Applying Newton's second law to the rightward acceleration a of both blocks (remembering that the equal and opposite forces exerted between the blocks cancel one another out), we obtain

$$a = \frac{F}{m_A + m_B},$$

where m_A is the mass of block A. It follows that

$$F > \frac{(m_A + m_B)\, g}{\mu}.$$

Since $m_A = 75\,\text{kg}$, $m_B = 15\,\text{kg}$, and $\mu = 0.45$, we have

$$F > \frac{(75 + 15) \times 9.81}{0.45} = 1.962 \times 10^3\,\text{N}.$$

5 Conservation of energy

5.1 Introduction

Nowadays, the *conservation of energy* is undoubtedly the single most important idea in physics. Strangely enough, although the basic idea of energy conservation was familiar to scientists from the time of Newton onwards, this crucial concept only moved to centre-stage in physics in about 1850 (*i.e.*, when scientists first realized that heat was a form of energy).

According to the ideas of modern physics, *energy* is the substance from which all things in the Universe are made up. Energy can take many different forms: *e.g.*, potential energy, kinetic energy, electrical energy, thermal energy, chemical energy, nuclear energy, *etc*. In fact, everything that we observe in the world around us represents one of the multitudinous manifestations of energy. Now, there exist processes in the Universe which transform energy from one form into another: *e.g.*, mechanical processes (which are the focus of this course), thermal processes, electrical processes, nuclear processes, *etc*. However, all of these processes leave the *total* amount of energy in the Universe *invariant*. In other words, whenever, and however, energy is transformed from one form into another, it is always *conserved*. For a closed system (*i.e.*, a system which does not exchange energy with the rest of the Universe), the above law of universal energy conservation implies that the total energy of the system in question must remain constant in time.

5.2 Energy conservation during free-fall

Consider a mass m which is falling vertically under the influence of gravity. We already know how to analyze the motion of such a mass. Let us employ this knowledge to search for an expression for the conserved energy during this process. (*N.B.*, This is clearly an example of a closed system, involving only the mass and the gravitational field.) The physics of free-fall under gravity is summarized by the three equations (2.14)–(2.16). Let us examine the last of these equations:

$v^2 = v_0^2 - 2\,g\,s$. Suppose that the mass falls from height h_1 to h_2, its initial velocity is v_1, and its final velocity is v_2. It follows that the net vertical displacement of the mass is $s = h_2 - h_1$. Moreover, $v_0 = v_1$ and $v = v_2$. Hence, the previous expression can be rearranged to give

$$\frac{1}{2}\,m\,v_1^2 + m\,g\,h_1 = \frac{1}{2}\,m\,v_2^2 + m\,g\,h_2. \tag{5.1}$$

The above equation clearly represents a conservation law, of some description, since the left-hand side only contains quantities evaluated at the initial height, whereas the right-hand side only contains quantities evaluated at the final height. In order to clarify the meaning of Eq. (5.1), let us define the *kinetic energy* of the mass,

$$K = \frac{1}{2}\,m\,v^2, \tag{5.2}$$

and the *gravitational potential energy* of the mass,

$$U = m\,g\,h. \tag{5.3}$$

Note that kinetic energy represents energy the mass possesses by virtue of its *motion*. Likewise, potential energy represents energy the mass possesses by virtue of its *position*. It follows that Eq. (5.1) can be written

$$E = K + U = \text{constant}. \tag{5.4}$$

Here, E is the *total* energy of the mass: *i.e.*, the sum of its kinetic and potential energies. It is clear that E is a conserved quantity: *i.e.*, although the kinetic and potential energies of the mass vary as it falls, its total energy remains the same.

Incidentally, the expressions (5.2) and (5.3) for kinetic and gravitational potential energy, respectively, are quite general, and do not just apply to free-fall under gravity. The mks unit of energy is called the *joule* (symbol J). In fact, 1 joule is equivalent to 1 kilogram meter-squared per second-squared, or 1 newton-meter. Note that all forms of energy are measured in the *same* units (otherwise the idea of energy conservation would make no sense).

One of the most important lessons which students learn during their studies is that there are generally many different paths to the same result in physics. Now, we have already analyzed free-fall under gravity using Newton's laws of motion. However, it is illuminating to re-examine this problem from the point of view of energy conservation. Suppose that a mass m is dropped from rest and falls a distance h. What is the final velocity v of the mass? Well, according to Eq. (5.1), if energy is conserved then

$$\Delta K = -\Delta U : \tag{5.5}$$

i.e., any increase in the kinetic energy of the mass must be offset by a corresponding decrease in its potential energy. Now, the change in potential energy of the mass is simply $\Delta U = m\,g\,s = -m\,g\,h$, where $s = -h$ is its net vertical displacement. The change in kinetic energy is simply $\Delta K = (1/2)\,m\,v^2$, where v is the final velocity. This follows because the initial kinetic energy of the mass is zero (since it is initially at rest). Hence, the above expression yields

$$\frac{1}{2}\,m\,v^2 = m\,g\,h, \tag{5.6}$$

or

$$v = \sqrt{2\,g\,h}. \tag{5.7}$$

Suppose that the same mass is thrown upwards with initial velocity v. What is the maximum height h to which it rises? Well, it is clear from Eq. (5.3) that as the mass rises its potential energy *increases*. It, therefore, follows from energy conservation that its kinetic energy must *decrease* with height. Note, however, from Eq. (5.2), that kinetic energy can never be negative (since it is the product of the two positive definite quantities, m and $v^2/2$). Hence, once the mass has risen to a height h which is such that its kinetic energy is reduced to *zero* it can rise no further, and must, presumably, start to fall. The change in potential energy of the mass in moving from its initial height to its maximum height is $m\,g\,h$. The corresponding change in kinetic energy is $-(1/2)\,m\,v^2$; since $(1/2)\,m\,v^2$ is the initial kinetic energy, and the final kinetic energy is zero. It follows from Eq. (5.5) that $-(1/2)\,m\,v^2 = -m\,g\,h$, which can be rearranged to give

$$h = \frac{v^2}{2\,g}. \tag{5.8}$$

It should be noted that the idea of energy conservation—although extremely useful—is *not* a replacement for Newton's laws of motion. For instance, in the previous example, there is no way in which we can deduce *how long* it takes the mass to rise to its maximum height from energy conservation alone—this information can only come from the direct application of Newton's laws.

5.3 Work

We have seen that when a mass free-falls under the influence of gravity some of its kinetic energy is transformed into potential energy, or *vice versa*. Let us now investigate, in detail, how this transformation is effected. The mass falls because it is subject to a downwards gravitational force of magnitude $m\,g$. It stands to reason, therefore, that the transformation of kinetic into potential energy is a direct consequence of the action of this force.

This is, perhaps, an appropriate point at which to note that the concept of gravitational potential energy—although extremely useful—is, strictly speaking, *fictitious*. To be more exact, the potential energy of a body is not an intrinsic property of that body (unlike its kinetic energy). In fact, the gravitational potential energy of a given body is stored in the *gravitational field* which surrounds it. Thus, when the body rises, and its potential energy consequently increases by an amount ΔU; in reality, it is the energy of the gravitational field surrounding the body which increases by this amount. Of course, the increase in energy of the gravitational field is offset by a corresponding decrease in the body's kinetic energy. Thus, when we speak of a body's kinetic energy being transformed into potential energy, we are really talking about a flow of energy *from* the body *to* the surrounding gravitational field. This energy flow is mediated by the gravitational force exerted by the field on the body in question.

Incidentally, according to Einstein's general theory of relativity (1917), the gravitational field of a mass consists of the local distortion that mass induces in the fabric of space-time. Fortunately, however, we do not need to understand general relativity in order to talk about gravitational fields or gravitational potential energy. All we need to know is that a gravitational field stores energy *without loss*: *i.e.*, if a given mass rises a certain distance, and, thereby, gives up a certain amount of energy to the surrounding gravitational field, then that field will return this energy to the mass—without loss—if the mass falls by the same distance. In physics, we term such a field a *conservative field* (see later).

Suppose that a mass m falls a distance h. During this process, the energy of the gravitational field decreases by a certain amount (*i.e.*, the fictitious potential energy of the mass decreases by a certain amount), and the body's kinetic energy increases by a corresponding amount. This transfer of energy, from the field to the mass, is, presumably, mediated by the gravitational force $-m\,g$ (the minus sign indicates that the force is directed downwards) acting on the mass. In fact, given that $U = m\,g\,h$, it follows from Eq. (5.5) that

$$\Delta K = f\,\Delta h. \tag{5.9}$$

In other words, the amount of energy transferred to the mass (*i.e.*, the increase in the mass's kinetic energy) is equal to the product of the force acting on the mass and the distance moved by the mass *in the direction of that force*.

In physics, we generally refer to the amount of energy transferred to a body, when a force acts upon it, as the amount of *work* W performed by that force on the body in question. It follows from Eq. (5.9) that when a gravitational force f acts on a body, causing it to displace a distance x *in the direction of that force*, then the net work done on the body is

$$W = f\,x. \tag{5.10}$$

It turns out that this equation is quite general, and does not just apply to gravitational forces. If W is positive then energy is transferred to the body, and its intrinsic energy consequently increases by an amount W. This situation occurs whenever a body moves in the *same* direction as the force acting upon it. Likewise, if W is negative then energy is transferred from the body, and its intrinsic energy consequently decreases by an amount $|W|$. This situation occurs whenever a body moves in the *opposite* direction to the force acting upon it. Since an amount of work is equivalent to a transfer of energy, the mks unit of work is the same as the mks unit of energy: namely, the joule.

In deriving equation (5.10), we have made two assumptions which are not universally valid. Firstly, we have assumed that the motion of the body upon which the force acts is both 1-dimensional and parallel to the line of action of the force. Secondly, we have assumed that the force does not vary with position. Let us attempt to relax these two assumptions, so as to obtain an expression for the work W done by a general force $\mathbf{f}$.

Let us start by relaxing the first assumption. Suppose, for the sake of argument, that we have a mass m which moves under gravity in 2-dimensions. Let us adopt the coordinate system shown in Fig. 35, with z representing vertical distance, and x representing horizontal distance. The vector acceleration of the mass is simply $\mathbf{a} = (0, -g)$. Here, we are neglecting the redundant y-component, for the sake of simplicity. The physics of motion under gravity in more than 1-dimension is summarized by the three equations (3.35)–(3.37). Let us examine the last of these equations:

$$v^2 = v_0^2 + 2\,\mathbf{a}\cdot\mathbf{s}. \tag{5.11}$$

Here, v_0 is the speed at $t = 0$, v is the speed at $t = t$, and $\mathbf{s} = (\Delta x,\, \Delta z)$ is the net displacement of the mass during this time interval. Recalling the definition of a scalar product [*i.e.*, $\mathbf{a}\cdot\mathbf{b} = (a_x\, b_x + a_y\, b_y + a_z\, b_z)$], the above equation can be rearranged to give

$$\frac{1}{2}\,m\,v^2 - \frac{1}{2}\,m\,v_0^2 = -m\,g\,\Delta z. \tag{5.12}$$

Since the right-hand side of the above expression is manifestly the increase in the kinetic energy of the mass between times $t = 0$ and $t = t$, the left-hand side must equal the decrease in the mass's potential energy during the same time interval. Hence, we arrive at the following expression for the gravitational potential energy of the mass:

$$U = m\,g\,z. \tag{5.13}$$

Of course, this expression is entirely equivalent to our previous expression for gravitational potential energy, Eq. (5.3). The above expression merely makes manifest a point which should have been obvious anyway: namely, that the gravitational potential energy of a mass only depends on its height above the ground, and is quite independent of its horizontal displacement.

Let us now try to relate the flow of energy between the gravitational field and the mass to the action of the gravitational force, $\mathbf{f} = (0, -m\,g)$. Equation (5.12)

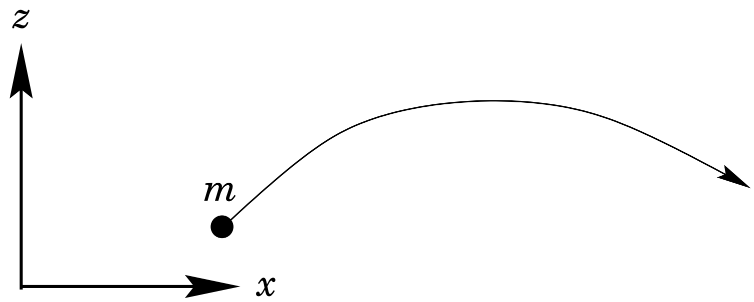


Figure 35: *Coordinate system for 2-dimensional motion under gravity*

can be rewritten

$$\Delta K = W = \mathbf{f}\cdot \mathbf{s}. \tag{5.14}$$

In other words, the work W done by the force $\mathbf{f}$ is equal to the scalar product of $\mathbf{f}$ and the vector displacement $\mathbf{s}$ of the body upon which the force acts. It turns out that this result is quite general, and does not just apply to gravitational forces.

Figure 36 is a visualization of the definition (5.14). The work W performed by a force $\mathbf{f}$ when the object upon which it acts is subject to a displacement $\mathbf{s}$ is

$$W = |\mathbf{f}|\ |\mathbf{s}|\ \cos\theta. \tag{5.15}$$

where θ is the angle subtended between the directions of $\mathbf{f}$ and $\mathbf{s}$. In other words, the work performed is the product of the magnitude of the force, $|\mathbf{f}|$, and the displacement of the object *in the direction of that force*, $|\mathbf{s}|\cos\theta$. It follows that any component of the displacement in a direction perpendicular to the force generates zero work. Moreover, if the displacement is entirely perpendicular to the direction of the force (*i.e.*, if $\theta = 90^\circ$) then no work is performed, irrespective of the nature of the force. As before, if the displacement has a component in the same direction as the force (*i.e.*, if $\theta < 90^\circ$) then positive work is performed Likewise, if the displacement has a component in the opposite direction to the force (*i.e.*, if $\theta > 90^\circ$) then negative work is performed.

Suppose, now, that an object is subject to a force $\mathbf{f}$ which varies with position. What is the total work done by the force when the object moves along some general trajectory in space between points A and B (say)? See Fig. 37. Well, one way in which we could approach this problem would be to approximate the trajectory as a series of N straight-line segments, as shown in Fig. 38. Suppose

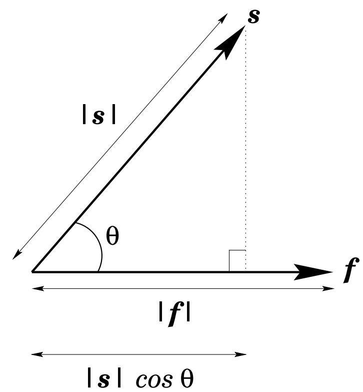


Figure 36: *Definition of work*

that the vector displacement of the ith segment is $\Delta\mathbf{r}_i$. Suppose, further, that N is sufficiently large that the force $\mathbf{f}$ does not vary much along each segment. In fact, let the average force along the ith segment be $\mathbf{f}_i$. We shall assume that formula (5.14)—which is valid for constant forces and straight-line displacements—holds good for each segment. It follows that the net work done on the body, as it moves from point A to point B, is approximately

$$W \simeq \sum_{i=1}^{N} \mathbf{f}_i \cdot \Delta\mathbf{r}_i. \tag{5.16}$$

We can always improve the level of our approximation by increasing the number N of the straight-line segments which we use to approximate the body's trajectory between points A and B. In fact, if we take the limit $N \to \infty$ then the above expression becomes exact:

$$W = \lim_{N\to\infty} \sum_{i=1}^{N} \mathbf{f}_i \cdot \Delta\mathbf{r}_i = \int_A^B \mathbf{f}(\mathbf{r}) \cdot d\mathbf{r}. \tag{5.17}$$

Here, $\mathbf{r}$ measures vector displacement from the origin of our coordinate system, and the mathematical construct $\int_A^B \mathbf{f}(\mathbf{r}) \cdot d\mathbf{r}$ is termed a *line-integral*.

The meaning of Eq. (5.17) becomes a lot clearer if we restrict our attention to 1-dimensional motion. Suppose, therefore, that an object moves in 1-dimension, with displacement x, and is subject to a varying force $f(x)$ (directed along the x-axis). What is the work done by this force when the object moves from x_A

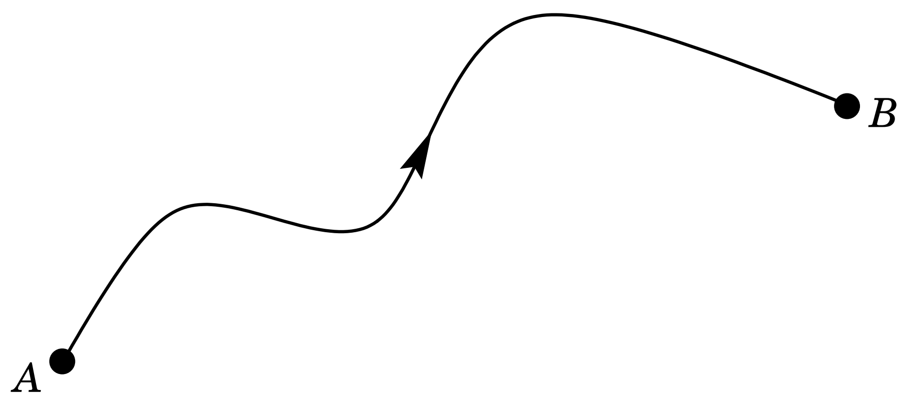


Figure 37: *Possible trajectory of an object in a variable force-field*

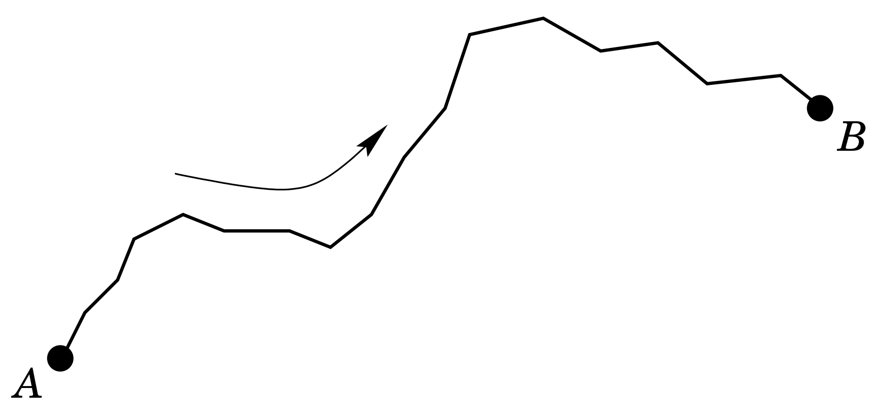


Figure 38: *Approximation to the previous trajectory using straight-line segments*

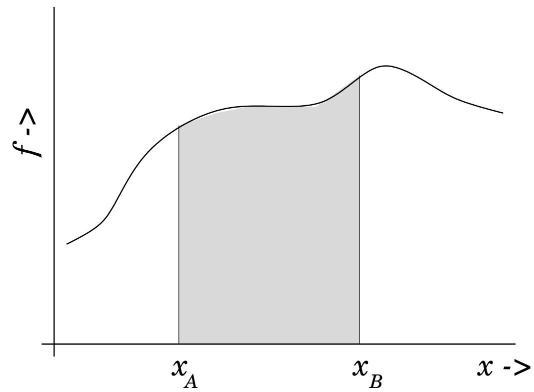


Figure 39: *Work performed by a 1-dimensional force*

to x_B? Well, a straight-forward application of Eq. (5.17) [with $\mathbf{f} = (f, 0, 0)$ and $d\mathbf{r} = (dx, 0, 0)$] yields

$$W = \int_{x_A}^{x_B} f(x)\, dx. \tag{5.18}$$

In other words, the net work done by the force as the object moves from displacement x_A to x_B is simply the area under the $f(x)$ curve between these two points, as illustrated in Fig. 39.

Let us, finally, round-off this discussion by re-deriving the so-called *work-energy theorem*, Eq. (5.14), in 1-dimension, allowing for a non-constant force. According to Newton's second law of motion,

$$f = m\,\frac{d^2x}{dt^2}. \tag{5.19}$$

Combining Eqs. (5.18) and (5.19), we obtain

$$W = \int_{x_A}^{x_B} m\,\frac{d^2x}{dt^2}\, dx = \int_{t_A}^{t_B} m\,\frac{d^2x}{dt^2}\frac{dx}{dt}\, dt = \int_{t_A}^{t_B} \frac{d}{dt}\left[\frac{m}{2}\left(\frac{dx}{dt}\right)^2\right] dt, \tag{5.20}$$

where $x(t_A) = x_A$ and $x(t_B) = x_B$. It follows that

$$W = \frac{1}{2}\, m\, v_B^2 - \frac{1}{2}\, m\, v_A^2 = \Delta K, \tag{5.21}$$

where $v_A = (dx/dt)_{t_A}$ and $v_B = (dx/dt)_{t_B}$. Thus, the net work performed on a body by a non-uniform force, as it moves from point A to point B, is equal to the net increase in that body's kinetic energy between these two points. This result is completely general (at least, for conservative force-fields—see later), and does not just apply to 1-dimensional motion.

Suppose, finally, that an object is subject to more than one force. How do we calculate the net work W performed by all these forces as the object moves from point A to point B? One approach would be to calculate the work done by each force, taken in isolation, and then to sum the results. In other words, defining

$$W_i = \int_A^B \mathbf{f}_i(\mathbf{r})\cdot d\mathbf{r} \tag{5.22}$$

as the work done by the ith force, the net work is given by

$$W = \sum_i W_i. \tag{5.23}$$

An alternative approach would be to take the vector sum of all the forces to find the resultant force,

$$\mathbf{f} = \sum_i \mathbf{f}_i, \tag{5.24}$$

and then to calculate the work done by the resultant force:

$$W = \int_A^B \mathbf{f}(\mathbf{r})\cdot d\mathbf{r}. \tag{5.25}$$

It should, hopefully, be clear that these two approaches are entirely equivalent.

5.4 Conservative and non-conservative force-fields

Suppose that a non-uniform force-field $\mathbf{f}(\mathbf{r})$ acts upon an object which moves along a curved trajectory, labeled path 1, from point A to point B. See Fig. 40. As we have seen, the work W_1 performed by the force-field on the object can be written as a line-integral along this trajectory:

$$W_1 = \int_{A\to B:\text{path}1} \mathbf{f}\cdot d\mathbf{r}. \tag{5.26}$$

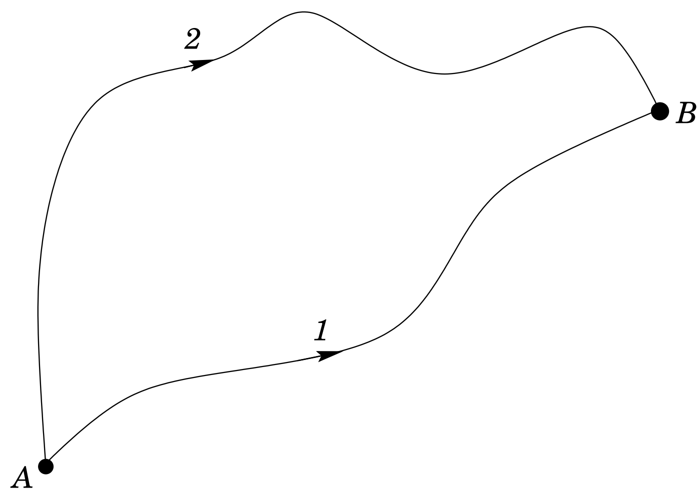


Figure 40: *Two alternative paths between points* A *and* B

Suppose that the same object moves along a different trajectory, labeled path 2, between the same two points. In this case, the work W_2 performed by the force-field is

$$W_2 = \int_{A\rightarrow B:\text{path2}} \mathbf{f}\cdot d\mathbf{r}. \tag{5.27}$$

Basically, there are two possibilities. Firstly, the line-integrals (5.26) and (5.27) might depend on the end points, A and B, but *not* on the path taken between them, in which case $W_1 = W_2$. Secondly, the line-integrals (5.26) and (5.27) might depend both on the end points, A and B, and the path taken between them, in which case $W_1 \neq W_2$ (in general). The first possibility corresponds to what physicists term a *conservative* force-field, whereas the second possibility corresponds to a *non-conservative* force-field.

What is the *physical* distinction between a conservative and a non-conservative force-field? Well, the easiest way of answering this question is to slightly modify the problem discussed above. Suppose, now, that the object moves from point A to point B along path 1, and then from point B back to point A along path 2. What is the total work done on the object by the force-field as it executes this closed circuit? Incidentally, one fact which should be clear from the definition of a line-integral is that if we simply reverse the path of a given integral then the value of that integral picks up a minus sign: in other words,

$$\int_A^B \mathbf{f}\cdot d\mathbf{r} = -\int_B^A \mathbf{f}\cdot d\mathbf{r}, \tag{5.28}$$

where it is understood that both the above integrals are taken in *opposite* directions along the *same* path. Recall that conventional 1-dimensional integrals obey an analogous rule: *i.e.*, if we swap the limits of integration then the integral picks up a minus sign. It follows that the total work done on the object as it executes the circuit is simply

$$\Delta W = W_1 - W_2, \tag{5.29}$$

where W_1 and W_2 are defined in Eqs. (5.26) and (5.27), respectively. There is a minus sign in front of W_2 because we are moving from point B to point A, instead of the other way around. For the case of a conservative field, we have $W_1 = W_2$. Hence, we conclude that

$$\Delta W = 0. \tag{5.30}$$

In other words, the net work done by a conservative field on an object taken around a closed loop is *zero*. This is just another way of saying that a conservative field stores energy without loss: *i.e.*, if an object gives up a certain amount of energy to a conservative field in traveling from point A to point B, then the field returns this energy to the object—without loss—when it travels back to point B. For the case of a non-conservative field, $W_1 \neq W_2$. Hence, we conclude that

$$\Delta W \neq 0. \tag{5.31}$$

In other words, the net work done by a non-conservative field on an object taken around a closed loop is non-zero. In practice, the net work is invariably *negative*. This is just another way of saying that a non-conservative field *dissipates energy*: *i.e.*, if an object gives up a certain amount of energy to a non-conservative field in traveling from point A to point B, then the field only returns part, or, perhaps, none, of this energy to the object when it travels back to point B. The remainder is usually dissipated as heat.

What are typical examples of conservative and non-conservative fields? Well, a gravitational field is probably the most well-known example of a conservative field (see later). A typical example of a non-conservative field might consist of

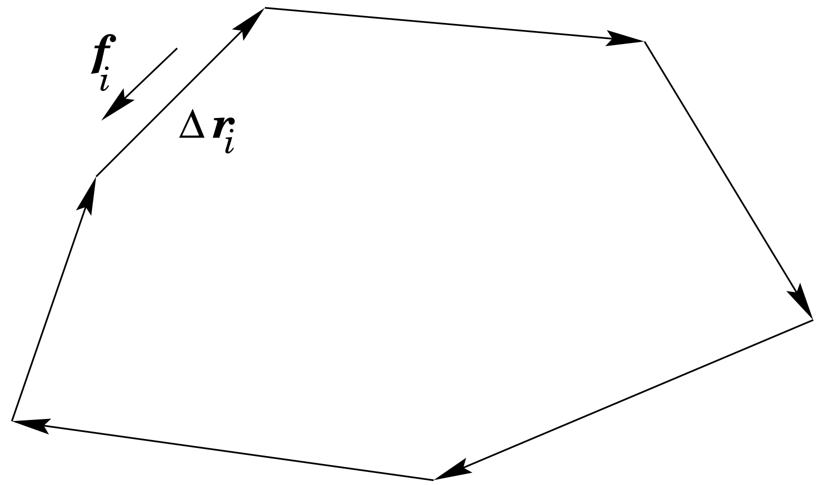


Figure 41: *Closed circuit over a rough horizontal surface*

an object moving over a rough horizontal surface. Suppose, for the sake of simplicity, that the object executes a closed circuit on the surface which is made up entirely of straight-line segments, as shown in Fig. 41. Let $\Delta\mathbf{r}_i$ represent the vector displacement of the ith leg of this circuit. Suppose that the frictional force acting on the object as it executes this leg is $\mathbf{f}_i$. One thing that we know about a frictional force is that it is always directed in the opposite direction to the instantaneous direction of motion of the object upon which it acts. Hence, $\mathbf{f}_i \propto -\Delta\mathbf{r}_i$. It follows that $\mathbf{f}_i\cdot\Delta\mathbf{r}_i = -|\mathbf{f}_i|\,|\Delta\mathbf{r}_i|$. Thus, the net work performed by the frictional force on the object, as it executes the circuit, is given by

$$\Delta W = \sum_i \mathbf{f}_i\cdot\Delta\mathbf{r}_i = -\sum_i |\mathbf{f}_i|\,|\Delta\mathbf{r}_i| < 0. \tag{5.32}$$

The fact that the net work is negative indicates that the frictional force continually drains energy from the object as it moves over the surface. This energy is actually dissipated as heat (we all know that if we rub two rough surfaces together, sufficiently vigorously, then they will eventually heat up: this is how mankind first made fire) and is, therefore, lost to the system. (Generally speaking, the laws of *thermodynamics* forbid energy which has been converted into heat from being converted back to its original form.) Hence, friction is an example of a non-conservative force, because it dissipates energy rather than storing it.

5.5 Potential energy

Consider a body moving in a conservative force-field $\mathbf{f}(\mathbf{r})$. Let us arbitrarily pick some point O in this field. We can define a function $U(\mathbf{r})$ which possesses a unique value at every point in the field. The value of this function associated with some general point R is simply

$$U(R) = -\int_O^R \mathbf{f}\cdot d\mathbf{r}. \tag{5.33}$$

In other words, $U(R)$ is just the energy transferred to the field (*i.e.*, minus the work done by the field) when the body moves from point O to point R. Of course, the value of U at point O is zero: *i.e.*, $U(O) = 0$. Note that the above definition *uniquely* specifies $U(R)$, since the work done when a body moves between two points in a conservative force-field is *independent* of the path taken between these points. Furthermore, the above definition would make no sense in a non-conservative field, since the work done when a body moves between two points in such a field is dependent on the chosen path: hence, $U(R)$ would have an infinite number of different values corresponding to the infinite number of different paths the body could take between points O and R.

According to the work-energy theorem,

$$\Delta K = \int_O^R \mathbf{f}\cdot d\mathbf{r}. \tag{5.34}$$

In other words, the net change in the kinetic energy of the body, as it moves from point O to point R, is equal to the work done on the body by the force-field during this process. However, comparing with Eq. (5.33), we can see that

$$\Delta K = U(O) - U(R) = -\Delta U. \tag{5.35}$$

In other words, the increase in the kinetic energy of the body, as it moves from point O to point R, is equal to the decrease in the function U evaluated between these same two points. Another way of putting this is

$$E = K + U = \text{constant}: \tag{5.36}$$

i.e., the sum of the kinetic energy and the function U remains constant as the body moves around in the force-field. It should be clear, by now, that the function U represents some form of *potential energy*.

The above discussion leads to the following important conclusions. Firstly, it should be possible to associate a potential energy (*i.e.*, an energy a body possesses by virtue of its position) with *any* conservative force-field. Secondly, any force-field for which we can define a potential energy must necessarily be conservative. For instance, the existence of gravitational potential energy is proof that gravitational fields are conservative. Thirdly, the concept of potential energy is meaningless in a non-conservative force-field (since the potential energy at a given point cannot be uniquely defined). Fourthly, potential energy is only defined to within an arbitrary additive constant. In other words, the point in space at which we set the potential energy to zero can be chosen at will. This implies that only *differences* in potential energies between different points in space have any physical significance. For instance, we have seen that the definition of gravitational potential energy is $U = m\,g\,z$, where z represents height above the ground. However, we could just as well write $U = m\,g\,(z - z_0)$, where z_0 is the height of some arbitrarily chosen reference point (*e.g.*, the top of Mount Everest, or the bottom of the Dead Sea). Fifthly, the difference in potential energy between two points represents the net energy transferred to the associated force-field when a body moves between these two points. In other words, potential energy is not, strictly speaking, a property of the body—instead, it is a property of the force-field within which the body moves.

5.6 Hooke's law

Consider a mass m which slides over a horizontal frictionless surface. Suppose that the mass is attached to a light horizontal spring whose other end is anchored to an immovable object. See Fig. 42. Let x be the extension of the spring: *i.e.*, the difference between the spring's actual length and its unstretched length. Obviously, x can also be used as a coordinate to determine the horizontal displacement of the mass. According to Hooke's law, the force f that the spring exerts on

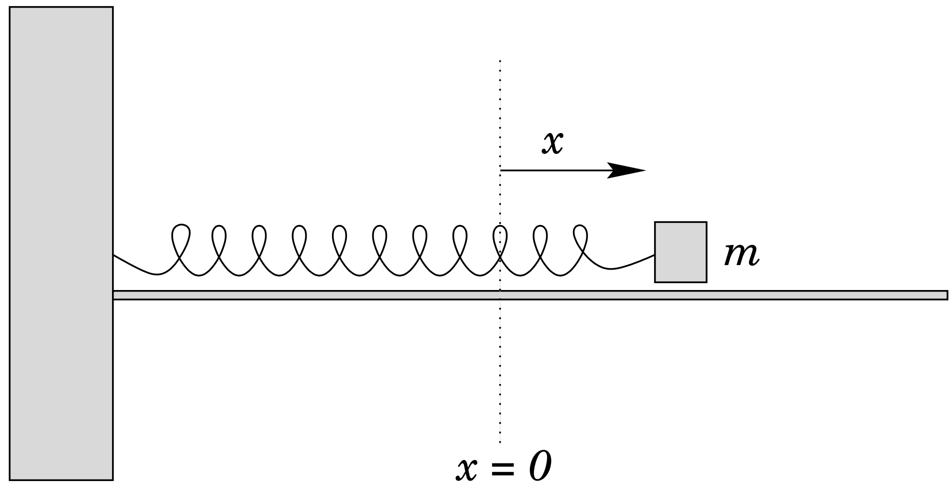


Figure 42: *Mass on a spring*

the mass is directly proportional to its extension, and always acts to reduce this extension. Hence, we can write

$$f = -k\,x, \tag{5.37}$$

where the positive quantity k is called the *force constant* of the spring. Note that the minus sign in the above equation ensures that the force always acts to reduce the spring's extension: *e.g.*, if the extension is positive then the force acts to the left, so as to shorten the spring.

According to Eq. (5.18), the work performed by the spring force on the mass as it moves from displacement x_A to x_B is

$$W = \int_{x_A}^{x_B} f(x)\,dx = -k\int_{x_A}^{x_B} x\,dx = -\left[\frac{1}{2}\,k\,x_B^2 - \frac{1}{2}\,k\,x_A^2\right]. \tag{5.38}$$

Note that the right-hand side of the above expression consists of the difference between two factors: the first only depends on the final state of the mass, whereas the second only depends on its initial state. This is a sure sign that it is possible to associate a *potential energy* with the spring force. Equation (5.33), which is the basic definition of potential energy, yields

$$U(x_B) - U(x_A) = -\int_{x_A}^{x_B} f(x)\,dx = \frac{1}{2}\,k\,x_B^2 - \frac{1}{2}\,k\,x_A^2. \tag{5.39}$$

Hence, the potential energy of the mass takes the form

$$U(x) = \frac{1}{2} k x^2. \tag{5.40}$$

Note that the above potential energy actually represents energy stored *by the spring*—in the form of mechanical stresses—when it is either stretched or compressed. Incidentally, this energy must be stored *without loss*, otherwise the concept of potential energy would be meaningless. It follows that the spring force is another example of a conservative force.

It is reasonable to suppose that the form of the spring potential energy is somehow related to the form of the spring force. Let us now explicitly investigate this relationship. If we let $x_B \to x$ and $x_A \to 0$ then Eq. (5.39) gives

$$U(x) = -\int_0^x f(x')\, dx'. \tag{5.41}$$

We can differentiate this expression to obtain

$$f = -\frac{dU}{dx}. \tag{5.42}$$

Thus, in 1-dimension, a conservative force is equal to minus the derivative (with respect to displacement) of its associated potential energy. This is a quite general result. For the case of a spring force: $U = (1/2)\, k\, x^2$, so $f = -dU/dx = -k\, x$.

As is easily demonstrated, the 3-dimensional equivalent to Eq. (5.42) is

$$\mathbf{f} = -\left(\frac{\partial U}{\partial x}, \frac{\partial U}{\partial y}, \frac{\partial U}{\partial z}\right). \tag{5.43}$$

For example, we have seen that the gravitational potential energy of a mass m moving above the Earth's surface is $U = m\, g\, z$, where z measures height off the ground. It follows that the associated gravitational force is

$$\mathbf{f} = (0, 0, -m\, g). \tag{5.44}$$

In other words, the force is of magnitude $m\, g$, and is directed vertically downward.

The total energy of the mass shown in Fig. 42 is the sum of its kinetic and potential energies:

$$E = K + U = K + \frac{1}{2}\,k\,x^2. \tag{5.45}$$

Of course, E remains constant during the mass's motion. Hence, the above expression can be rearranged to give

$$K = E - \frac{1}{2}\,k\,x^2. \tag{5.46}$$

Since it is impossible for a kinetic energy to be negative, the above expression suggests that $|x|$ can never exceed the value

$$x_0 = \sqrt{\frac{2\,E}{k}}. \tag{5.47}$$

Here, x_0 is termed the *amplitude* of the mass's motion. Note that when x attains its maximum value x_0, or its minimum value $-x_0$, the kinetic energy is momentarily zero (*i.e.*, $K = 0$).

5.7 Motion in a general 1-dimensional potential

Suppose that the curve $U(x)$ in Fig. 43 represents the potential energy of some mass m moving in a 1-dimensional conservative force-field. For instance, $U(x)$ might represent the gravitational potential energy of a cyclist freewheeling in a hilly region. Note that we have set the potential energy at infinity to zero. This is a useful, and quite common, convention (recall that potential energy is undefined to within an arbitrary additive constant). What can we deduce about the motion of the mass in this potential?

Well, we know that the total energy, E—which is the sum of the kinetic energy, K, and the potential energy, U—is a *constant* of the motion. Hence, we can write

$$K(x) = E - U(x). \tag{5.48}$$

Now, we also know that a kinetic energy can never be negative, so the above expression tells us that the motion of the mass is restricted to the region (or

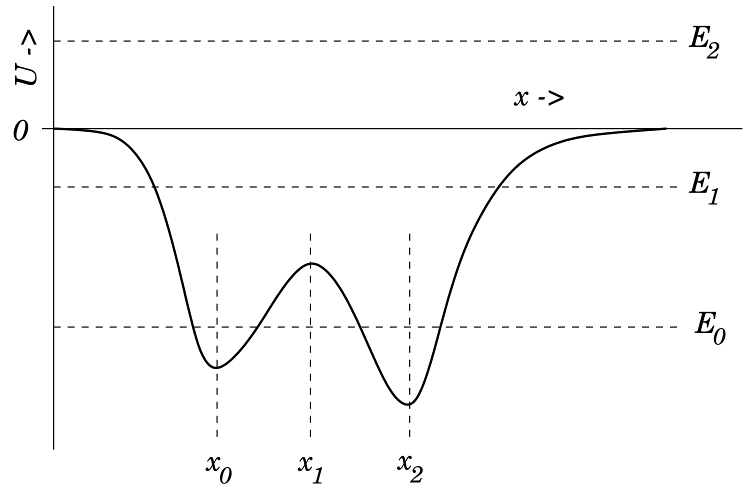


Figure 43: *General 1-dimensional potential*

regions) in which the potential energy curve $U(x)$ falls below the value E. This idea is illustrated in Fig. 43. Suppose that the total energy of the system is E_0. It is clear, from the figure, that the mass is trapped inside one or other of the two dips in the potential—these dips are generally referred to as *potential wells*. Suppose that we now raise the energy to E_1. In this case, the mass is free to enter or leave each of the potential wells, but its motion is still *bounded* to some extent, since it clearly cannot move off to infinity. Finally, let us raise the energy to E_2. Now the mass is *unbounded*: *i.e.*, it can move off to infinity. In systems in which it makes sense to adopt the convention that the potential energy at infinity is zero, bounded systems are characterized by $E < 0$, whereas unbounded systems are characterized by $E > 0$.

The above discussion suggests that the motion of a mass moving in a potential generally becomes less bounded as the total energy E of the system increases. Conversely, we would expect the motion to become more bounded as E decreases. In fact, if the energy becomes sufficiently small, it appears likely that the system will settle down in some *equilibrium state* in which the mass is stationary. Let us try to identify any prospective equilibrium states in Fig. 43. If the mass remains stationary then it must be subject to zero force (otherwise it would accelerate). Hence, according to Eq. (5.42), an equilibrium state is characterized by

$$\frac{dU}{dx} = 0. \tag{5.49}$$

In other words, a equilibrium state corresponds to either a *maximum* or a *minimum* of the potential energy curve $U(x)$. It can be seen that the $U(x)$ curve shown in Fig. 43 has three associated equilibrium states: these are located at $x = x_0$, $x = x_1$, and $x = x_2$.

Let us now make a distinction between *stable equilibrium* points and *unstable equilibrium* points. When the system is slightly perturbed from a stable equilibrium point then the resultant force f should always be such as to attempt to return the system to this point. In other words, if $x = x_0$ is an equilibrium point, then we require

$$\left.\frac{df}{dx}\right|_{x_0} < 0 \tag{5.50}$$

for stability: *i.e.*, if the system is perturbed to the right, so that $x - x_0 > 0$, then the force must act to the left, so that $f < 0$, and *vice versa*. Likewise, if

$$\left.\frac{df}{dx}\right|_{x_0} > 0 \tag{5.51}$$

then the equilibrium point $x = x_0$ is unstable. It follows, from Eq. (5.42), that stable equilibrium points are characterized by

$$\frac{d^2U}{dx^2} > 0. \tag{5.52}$$

In other words, a stable equilibrium point corresponds to a *minimum* of the potential energy curve $U(x)$. Likewise, an unstable equilibrium point corresponds to a *maximum* of the $U(x)$ curve. Hence, we conclude that $x = x_0$ and $x = x_2$ are stable equilibrium points, in Fig. 43, whereas $x = x_1$ is an unstable equilibrium point. Of course, this makes perfect sense if we think of $U(x)$ as a gravitational potential energy curve, in which case U is directly proportional to height. All we are saying is that it is easy to confine a low energy mass at the bottom of a valley, but very difficult to balance the same mass on the top of a hill (since any slight perturbation to the mass will cause it to fall down the hill). Note, finally, that if

$$\frac{dU}{dx} = \frac{d^2U}{dx^2} = 0 \tag{5.53}$$

at any point (or in any region) then we have what is known as a *neutral equilibrium* point. We can move the mass slightly off such a point and it will still remain

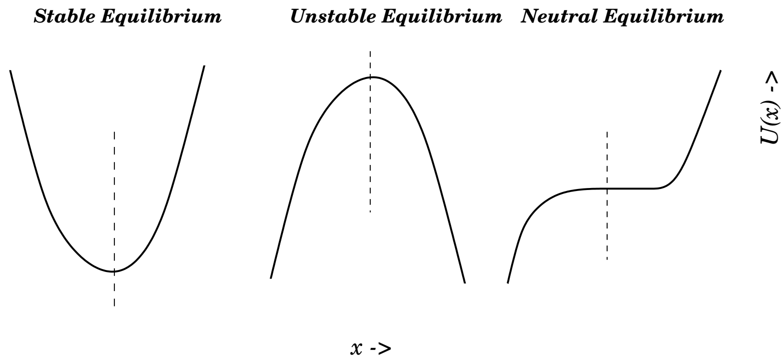


Figure 44: *Different types of equilibrium*

in equilibrium (*i.e.*, it will neither attempt to return to its initial state, nor will it continue to move). A neutral equilibrium point corresponds to a *flat spot* in a $U(x)$ curve. See Fig. 44.

5.8 Power

Suppose that an object moves in a general force-field $\mathbf{f}(\mathbf{r})$. We now know how to calculate how much energy flows from the force-field to the object as it moves along a given path between two points. Let us now consider the *rate* at which this energy flows. If dW is the amount of work that the force-field performs on the mass in a time interval dt then the rate of working is given by

$$P = \frac{dW}{dt}. \tag{5.54}$$

In other words, the rate of working—which is usually referred to as the *power*—is simply the time derivative of the work performed. Incidentally, the mks unit of power is called the *watt* (symbol W). In fact, 1 watt equals 1 kilogram meter-squared per second-cubed, or 1 joule per second.

Suppose that the object displaces by $d\mathbf{r}$ in the time interval dt. By definition, the amount of work done on the object during this time interval is given by

$$dW = \mathbf{f}\cdot d\mathbf{r}. \tag{5.55}$$

It follows from Eq. (5.54) that

$$P = \mathbf{f}\cdot\mathbf{v}, \tag{5.56}$$

where $\mathbf{v} = d\mathbf{r}/dt$ is the object's instantaneous velocity. Note that power can be positive or negative, depending on the relative directions of the vectors $\mathbf{f}$ and $\mathbf{v}$. If these two vectors are mutually perpendicular then the power is zero. For the case of 1-dimensional motion, the above expression reduces to

$$P = f\,v. \tag{5.57}$$

In other words, in 1-dimension, power simply equals force times velocity.

Worked example 5.1: Bucket lifted from a well

Question: A man lifts a 30 kg bucket from a well whose depth is 150 m. Assuming that the man lifts the bucket at a constant rate, how much work does he perform?

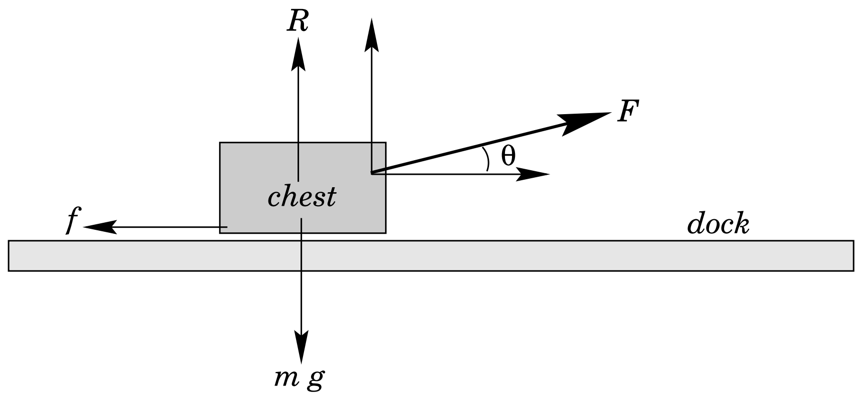
Answer: Let m be the mass of the bucket and h the depth of the well. The gravitational force f' acting on the bucket is of magnitude $m\,g$ and is directed vertically downwards. Hence, $f' = -m\,g$ (where upward is defined to be positive). The net upward displacement of the bucket is h. Hence, the work W' performed by the gravitational force is the product of the (constant) force and the displacement of the bucket along the line of action of that force:

$$W' = f'\,h = -m\,g\,h.$$

Note that W' is negative, which implies that the gravitational field surrounding the bucket gains energy as the bucket is lifted. In order to lift the bucket at a constant rate, the man must exert a force f on the bucket which balances (and very slightly exceeds) the force due to gravity. Hence, $f = -f'$. It follows that the work W done by the man is

$$W = f\,h = m\,g\,h = 30 \times 150 \times 9.81 = 4.415 \times 10^4\,\text{J}.$$

Note that the work is positive, which implies that the man expends energy whilst lifting the bucket. Of course, since $W = -W'$, the energy expended by the man equals the energy gained by the gravitational field.



Worked example 5.2: Dragging a treasure chest

Question: A pirate drags a 50 kg treasure chest over the rough surface of a dock by exerting a constant force of 95 N acting at an angle of 15° above the horizontal. The chest moves 6 m in a straight line, and the coefficient of kinetic friction between the chest and the dock is 0.15. How much work does the pirate perform? How much energy is dissipated as heat via friction? What is the final velocity of the chest?

Answer: Referring to the diagram, the force F exerted by the pirate can be resolved into a horizontal component $F\cos\theta$ and a vertical component $F\sin\theta$. Since the chest only moves horizontally, the vertical component of F performs zero work. The work W performed by the horizontal component is simply the magnitude of this component times the horizontal distance x moved by the chest:

$$W = F\cos\theta\, x = 95\times\cos 15^\circ\times 6 = 550.6\,\mathrm{J}.$$

The chest is subject to the following forces in the vertical direction: the downward force $m\,g$ due to gravity, the upward reaction force R due to the dock, and the upward component $F\sin\theta$ of the force exerted by the pirate. Since the chest does not accelerate in the vertical direction, these forces must balance. Hence,

$$R = m\,g - F\sin\theta = 50\times 9.81 - 95\times\sin 15^\circ = 465.9\,\mathrm{N}.$$

The frictional force f is the product of the coefficient of kinetic friction μ_k and the normal reaction R, so

$$f = \mu_k\,R = 0.15\times 465.9 = 69.89\,\mathrm{N}.$$

The work W' done by the frictional force is

$$W' = -f\,x = -69.89 \times 6 = -419.3\,\mathrm{J}.$$

Note that there is a minus sign in front of the f because the displacement of the chest is in the opposite direction to the frictional force. The fact that W' is negative indicates a loss of energy by the chest: this energy is dissipated as heat via friction. Hence, the dissipated energy is 419.3 J.

The final kinetic energy K of the chest (assuming that it is initially at rest) is the difference between the work W done by the pirate and the energy $-W'$ dissipated as heat. Hence,

$$K = W + W' = 550.6 - 419.3 = 131.3\,\mathrm{J}.$$

Since $K = (1/2)\,m\,v^2$, the final velocity of the chest is

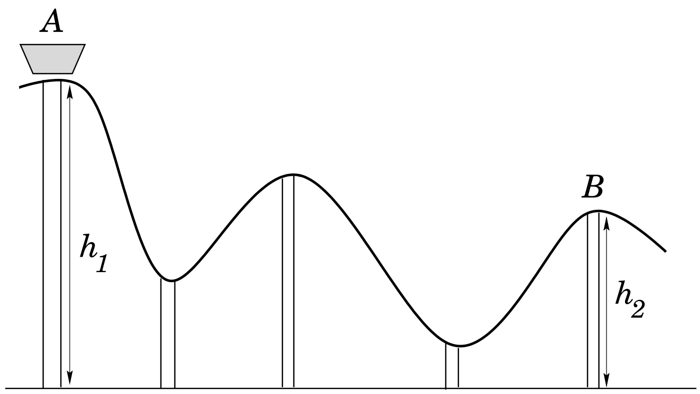
$$v = \sqrt{\frac{2\,K}{m}} = \sqrt{\frac{2 \times 131.3}{50}} = 2.29\,\mathrm{m/s}.$$

Worked example 5.3: Stretching a spring

Question: The force required to slowly stretch a spring varies from 0 N to 105 N as the spring is extended by 13 cm from its unstressed length. What is the force constant of the spring? What work is done in stretching the spring? Assume that the spring obeys Hooke's law.

Answer: The force f that the spring exerts on whatever is stretching it is $f = -k\,x$, where k is the force constant, and x is the extension of the spring. The minus sign indicates that the force acts in the opposite direction to the extension. Since the spring is stretched slowly, the force f' which must be exerted on it is (almost) equal and opposite to f. Hence, $f' = -f = k\,x$. We are told that $f' = 105\,\mathrm{N}$ when $x = 0.13\,\mathrm{m}$. It follows that

$$k = \frac{105}{0.13} = 807.7\,\mathrm{N/m}.$$



The work W' done by the external force in extending the spring from 0 to x is

$$W' = \int_0^x f'\,dx = k\int_0^x x\,dx = \frac{1}{2}\,k\,x^2.$$

Hence,

$$W' = 0.5 \times 807.7 \times 0.13^2 = 6.83\,\text{J}.$$

Worked example 5.4: Roller coaster ride

Question: A roller coaster cart of mass $m = 300\,\text{kg}$ starts at rest at point A, whose height off the ground is $h_1 = 25\,\text{m}$, and a little while later reaches point B, whose height off the ground is $h_2 = 7\,\text{m}$. What is the potential energy of the cart relative to the ground at point A? What is the speed of the cart at point B, neglecting the effect of friction?

Answer: The gravitational potential energy of the cart with respect to the ground at point A is

$$U_A = m\,g\,h_1 = 300 \times 9.81 \times 25 = 7.36 \times 10^4\,\text{J}.$$

Likewise, the potential energy of the cart at point B is

$$U_B = m\,g\,h_2 = 300 \times 9.81 \times 7 = 2.06 \times 10^4\,\text{J}.$$

Hence, the change in the cart's potential energy in moving from point A to point B is

$$\Delta U = U_B - U_A = 2.06 \times 10^4 - 7.36 \times 10^4 = -5.30 \times 10^4\,\text{J}.$$

By energy conservation, $\Delta K = -\Delta U$, where K represents kinetic energy. However, since the initial kinetic energy is zero, the change in kinetic energy ΔK is equivalent to the final kinetic energy K_B. Thus,

$$K_B = -\Delta U = 5.30 \times 10^4\,\text{J}.$$

Now, $K_B = (1/2)\,m\,v_B^2$, where v_B is the final speed. Hence,

$$v_B = \sqrt{\frac{2\,K_B}{m}} = \sqrt{\frac{2 \times 5.30 \times 10^4}{300}} = 18.8\,\text{m/s}.$$

Worked example 5.5: Sliding down a plane

Question: A block of mass $m = 3\,\text{kg}$ starts at rest at a height of $h = 43\,\text{cm}$ on a plane that has an angle of inclination of $\theta = 35^\circ$ with respect to the horizontal. The block slides down the plane, and, upon reaching the bottom, then slides along a horizontal surface. The coefficient of kinetic friction of the block on both surfaces is $\mu = 0.25$. How far does the block slide along the horizontal surface before coming to rest?

Answer: The normal reaction of the plane to the block's weight is

$$R = m\,g\,\cos\theta.$$

Hence, the frictional force acting on the block when it is sliding down the plane is

$$f = \mu\,R = 0.25 \times 3 \times 9.81 \times \cos 35^\circ = 6.03\,\text{N}.$$

The change in gravitational potential energy of the block as it slides down the plane is

$$\Delta U = -m\,g\,h = -3 \times 9.81 \times 0.43 = -12.65\,\text{J}.$$

The work W done on the block by the frictional force during this process is

$$W = -f\,x,$$

where $x = h/\sin\theta$ is the distance the block slides. The minus sign indicates that f acts in the opposite direction to the displacement of the block. Hence,

$$W = -\frac{6.03 \times 0.43}{\sin 35^\circ} = -4.52\,\mathrm{J}.$$

Now, by energy conservation, the kinetic energy K of the block at the bottom of the plane equals the decrease in the block's potential energy plus the amount of work done on the block:

$$K = -\Delta U + W = 12.65 - 4.52 = 8.13\,\mathrm{J}.$$

The frictional force acting on the block when it slides over the horizontal surface is

$$f' = \mu\, m\, g = 0.25 \times 3 \times 9.81 = 7.36\,\mathrm{N}.$$

The work done on the block as it slides a distance y over this surface is

$$W' = -f'\,y.$$

By energy conservation, the block comes to rest when the action of the frictional force has drained all of the kinetic energy from the block: *i.e.*, when $W' = -K$. It follows that

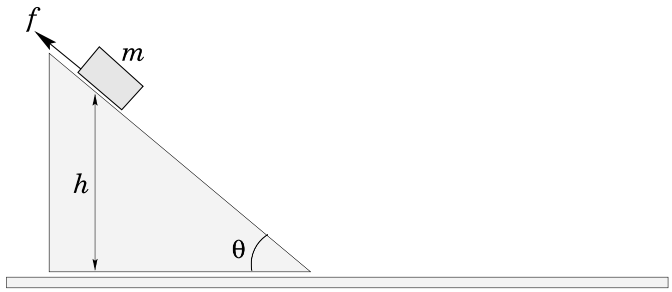
$$y = \frac{K}{f'} = \frac{8.13}{7.36} = 1.10\,\mathrm{m}.$$

Worked example 5.6: Driving up an incline

Question: A car of weight 3000 N possesses an engine whose maximum power output is 160 kW. The maximum speed of this car on a level road is 35 m/s. Assuming that the resistive force (due to a combination of friction and air resistance) remains constant, what is the car's maximum speed on an incline of 1 in 20 (*i.e.*, if θ is the angle of the incline with respect to the horizontal, then $\sin\theta = 1/20$)?

Answer: When the car is traveling on a level road at its maximum speed, v, then all of the power output, P, of its engine is used to overcome the power dissipated by the resistive force, f. Hence,

$$P = f\,v$$



where the left-hand side is the power output of the engine, and the right-hand side is the power dissipated by the resistive force (*i.e.*, minus the rate at which this force does work on the car). It follows that

$$f = \frac{P}{v} = \frac{160 \times 10^3}{35} = 4.57 \times 10^3 \,\text{N}.$$

When the car, whose weight is W, is traveling up an incline, whose angle with respect to the horizontal is θ, it is subject to the additional force $f' = W \sin\theta$, which acts to impede its motion. Of course, this force is just the component of the car's weight acting down the incline. Thus, the new power balance equation is written

$$P = f\,v' + W \sin\theta\, v',$$

where v' is the maximum velocity of the car up the incline. Here, the left-hand side represents the power output of the car, whereas the right-hand side represents the sum of the power dissipated by the resistive force and the power expended to overcome the component of the car's weight acting down the incline. It follows that

$$v' = \frac{P}{f + W \sin\theta} = \frac{160 \times 10^3}{4.57 \times 10^3 + 3000/20} = 33.90 \,\text{m/s}.$$

6 Conservation of momentum

6.1 Introduction

Up to now, we have only analyzed the behaviour of dynamical systems which consist of *single* point masses (*i.e.*, objects whose spatial extent is either negligible or plays no role in their motion) or arrangements of point masses which are constrained to move *together* because they are connected via inextensible cables. Let us now broaden our approach somewhat in order to take into account systems of point masses which exert forces on one another, but are not necessarily constrained to move together. The classic example of such a multi-component point mass system is one in which two (or more) freely moving masses *collide* with one another. The physical concept which plays the central role in the dynamics of multi-component point mass systems is the *conservation of momentum*.

6.2 Two-component systems

The simplest imaginable multi-component dynamical system consists of two point mass objects which are both constrained to move along the same straight-line. See Fig. 45. Let x_1 be the displacement of the first object, whose mass is m_1. Likewise, let x_2 be the displacement of the second object, whose mass is m_2. Suppose that the first object exerts a force f_{21} on the second object, whereas the second object exerts a force f_{12} on the first. From Newton's third law of motion, we have

$$f_{12} = -f_{21}. \tag{6.1}$$

Suppose, finally, that the first object is subject to an external force (*i.e.*, a force which originates *outside* the system) F_1, whilst the second object is subject to an external force F_2.

Applying Newton's second law of motion to each object in turn, we obtain

$$m_1\,\ddot{x}_1 = f_{12} + F_1, \tag{6.2}$$

$$m_2\,\ddot{x}_2 = f_{21} + F_2. \tag{6.3}$$

Here, $\dot{}$ is a convenient shorthand for d/dt. Likewise, $\ddot{}$ means d^2/dt^2.

At this point, it is helpful to introduce the concept of the *centre of mass*. The centre of mass is an imaginary point whose displacement x_{cm} is defined to be the *mass weighted average* of the displacements of the two objects which constitute the system. In other words,

$$x_{cm} = \frac{m_1\, x_1 + m_2\, x_2}{m_1 + m_2}. \tag{6.4}$$

Thus, if the two masses are equal then the centre of mass lies half way between them; if the second mass is three times larger than the first then the centre of mass lies three-quarters of the way along the line linking the first and second masses, respectively; if the second mass is much larger than the first then the centre of mass is almost coincident with the second mass; and so on.

Summing Eqs. (6.2) and (6.3), and then making use of Eqs. (6.1) and (6.4), we obtain

$$m_1\, \ddot{x}_1 + m_2\, \ddot{x}_2 = (m_1 + m_2)\, \ddot{x}_{cm} = F_1 + F_2. \tag{6.5}$$

Note that the *internal forces*, f_{12} and f_{21}, have canceled out. The physical significance of this equation becomes clearer if we write it in the following form:

$$M\, \ddot{x}_{cm} = F, \tag{6.6}$$

where $M = m_1 + m_2$ is the total mass of the system, and $F = F_1 + F_2$ is the net *external* force acting on the system. Thus, the motion of the centre of mass is equivalent to that which would occur if all the mass contained in the system were collected at the centre of mass, and this conglomerate mass were then acted upon by the net external force. In general, this suggests that the motion of the centre of mass is *simpler* than the motions of the component masses, m_1 and m_2.

Figure 45: *A 1-dimensional dynamical system consisting of two point mass objects*

This is particularly the case if the internal forces, f_{12} and f_{21}, are complicated in nature.

Suppose that there are *no* external forces acting on the system (*i.e.*, $F_1 = F_2 = 0$), or, equivalently, suppose that the sum of all the external forces is *zero* (*i.e.*, $F = F_1 + F_2 = 0$). In this case, according to Eq. (6.6), the motion of the centre of mass is governed by Newton's first law of motion: *i.e.*, it consists of uniform motion in a straight-line. Hence, in the absence of a net external force, the motion of the centre of mass is almost certainly *far simpler* than that of the component masses.

Now, the velocity of the centre of mass is written

$$v_{cm} = \dot{x}_{cm} = \frac{m_1\,\dot{x}_1 + m_2\,\dot{x}_2}{m_1 + m_2}. \tag{6.7}$$

We have seen that in the absence of external forces v_{cm} is a constant of the motion (*i.e.*, the centre of mass does not accelerate). It follows that, in this case,

$$m_1\,\dot{x}_1 + m_2\,\dot{x}_2 \tag{6.8}$$

is also a constant of the motion. Recall, however, from Sect. 4.3, that *momentum* is defined as the product of mass and velocity. Hence, the momentum of the first mass is written $p_1 = m_1\,\dot{x}_1$, whereas the momentum of the second mass takes the form $p_2 = m_2\,\dot{x}_2$. It follows that the above expression corresponds to the *total momentum* of the system:

$$P = p_1 + p_2. \tag{6.9}$$

Thus, the total momentum is a conserved quantity—provided there is no net external force acting on the system. This is true *irrespective* of the nature of the internal forces. More generally, Eq. (6.6) can be written

$$\frac{dP}{dt} = F. \tag{6.10}$$

In other words, the time derivative of the total momentum is equal to the net external force acting on the system—this is just Newton's second law of motion applied to the system as a whole.

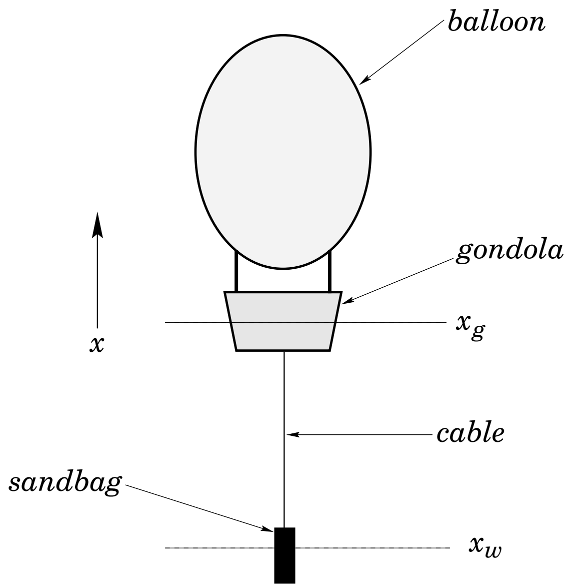


Figure 46: *An example two-component system*

Let us now try to apply some of the concepts discussed above. Consider the simple two-component system shown in Fig. 46. A gondola of mass m_g hangs from a hot-air balloon whose mass is negligible compared to that of the gondola. A sandbag of mass m_w is suspended from the gondola by means of a light inextensible cable. The system is in equilibrium. Suppose, for the sake of consistency with our other examples, that the x-axis runs vertically upwards. Let x_g be the height of the gondola, and x_w the height of the sandbag. Suppose that the upper end of the cable is attached to a winch inside the gondola, and that this winch is used to *slowly* shorten the cable, so that the sandbag is lifted upwards a distance Δx_w. The question is this: does the height of the gondola also change as the cable is reeled in? If so, by how much?

Let us identify all of the forces acting on the system shown in Fig. 46. The internal forces are the upward force exerted by the gondola on the sandbag, and the downward force exerted by the sandbag on the gondola. These forces are transmitted via the cable, and are equal and opposite (by Newton's third law of motion). The external forces are the net downward force due to the combined weight of the gondola and the sandbag, and the upward force due to the buoyancy of the balloon. Since the system is in equilibrium, these forces are equal and opposite (it is assumed that the cable is reeled in sufficiently slowly that the

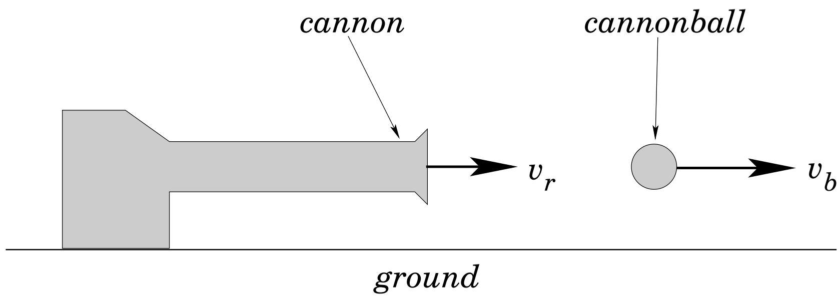


Figure 47: *Another example two-component system*

equilibrium is not upset). Hence, there is *zero* net external force acting on the system. It follows, from the previous discussion, that the centre of mass of the system is subject to Newton's first law. In particular, since the centre of mass is clearly *stationary* before the winch is turned on, it must *remain stationary* both during and after the period in which the winch is operated. Hence, the height of the centre of mass,

$$x_{cm} = \frac{m_g\, x_g + m_w\, x_w}{m_g + m_w}, \tag{6.11}$$

is a conserved quantity.

Suppose that the operation of the winch causes the height of the sandbag to change by Δx_w, and that of the gondola to simultaneously change by Δx_g. If x_{cm} is a conserved quantity, then we must have

$$0 = m_g\, \Delta x_g + m_w\, \Delta x_w, \tag{6.12}$$

or

$$\Delta x_g = -\frac{m_w}{m_g}\, \Delta x_w. \tag{6.13}$$

Thus, if the winch is used to *raise* the sandbag a distance Δx_w then the gondola is simultaneously pulled *downwards* a distance $(m_w/m_g)\, \Delta x_w$. It is clear that we could use a suspended sandbag as a mechanism for adjusting a hot-air balloon's altitude: the balloon descends as the sandbag is raised, and ascends as it is lowered.

Our next example is pictured in Fig. 47. Suppose that a cannon of mass M propels a cannonball of mass m horizontally with velocity v_b. What is the recoil velocity v_r of the cannon? Let us first identify all of the forces acting on the system. The internal forces are the force exerted by the cannon on the cannonball, as the cannon is fired, and the equal and opposite force exerted by the cannonball on the cannon. These forces are extremely large, but only last for a short instance in time: in physics, we call these *impulsive* forces. There are no external forces acting in the horizontal direction (which is the only direction that we are considering in this example). It follows that the total (horizontal) momentum P of the system is a conserved quantity. Prior to the firing of the cannon, the total momentum is *zero* (since momentum is mass times velocity, and nothing is initially moving). After the cannon is fired, the total momentum of the system takes the form

$$P = m\,v_b + M\,v_r. \tag{6.14}$$

Since P is a conserved quantity, we can set $P = 0$. Hence,

$$v_r = -\frac{m}{M}\,v_b. \tag{6.15}$$

Thus, the recoil velocity of the cannon is in the *opposite* direction to the velocity of the cannonball (hence, the minus sign in the above equation), and is of magnitude $(m/M)\,v_b$. Of course, if the cannon is far more massive that the cannonball (*i.e.*, $M \gg m$), which is usually the case, then the recoil velocity of the cannon is far smaller in magnitude than the velocity of the cannonball. Note, however, that the *momentum* of the cannon is equal in magnitude to that of the cannonball. It follows that it takes the same effort (*i.e.*, force applied for a certain period of time) to slow down and stop the cannon as it does to slow down and stop the cannonball.

6.3 Multi-component systems

Consider a system of N mutually interacting point mass objects which move in 3-dimensions. See Fig. 48. Let the ith object, whose mass is m_i, be located at vector displacement $\mathbf{r}_i$. Suppose that this object exerts a force $\mathbf{f}_{ji}$ on the jth object. By Newton's third law of motion, the force $\mathbf{f}_{ij}$ exerted by the jth object on the ith is given by

$$\mathbf{f}_{ij} = -\mathbf{f}_{ji}. \tag{6.16}$$

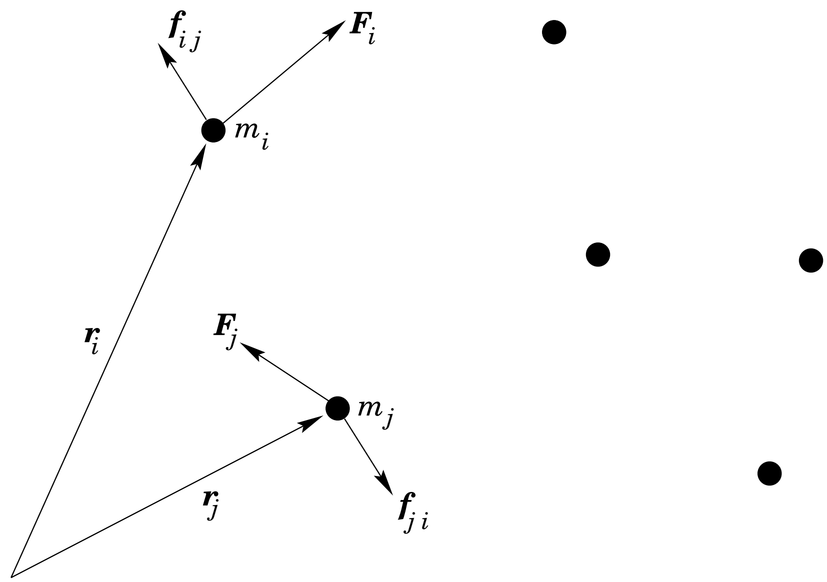


Figure 48: *A 3-dimensional dynamical system consisting of many point mass objects.*

Finally, suppose that the ith object is subject to an external force $\mathbf{F}_i$.

Newton's second law of motion applied to the ith object yields

$$m_i\,\ddot{\mathbf{r}}_i = \sum_{j=1,N}^{j\neq i} \mathbf{f}_{ij} + \mathbf{F}_i. \tag{6.17}$$

Note that the summation on the right-hand side of the above equation excludes the case $j = i$, since the ith object cannot exert a force on itself. Let us now take the above equation and sum it over all objects. We obtain

$$\sum_{i=1,N} m_i\,\ddot{\mathbf{r}}_i = \sum_{i,j=1,N}^{j\neq i} \mathbf{f}_{ij} + \sum_{i=1,N} \mathbf{F}_i. \tag{6.18}$$

Consider the sum over all internal forces: *i.e.*, the first term on the right-hand side. Each element of this sum—$\mathbf{f}_{ij}$, say—can be paired with another element—$\mathbf{f}_{ji}$, in this case—which is equal and opposite. In other words, the elements of the sum all cancel out in pairs. Thus, the net value of the sum is *zero*. It follows that the above equation can be written

$$M\,\ddot{\mathbf{r}}_{cm} = \mathbf{F}, \tag{6.19}$$

where $M = \sum_{i=1}^{N} m_i$ is the total mass, and $\mathbf{F} = \sum_{i=1}^{N} \mathbf{F}_i$ is the net external force. The quantity $\mathbf{r}_{cm}$ is the vector displacement of the centre of mass. As before, the centre of mass is an imaginary point whose coordinates are the mass weighted averages of the coordinates of the objects which constitute the system. Thus,

$$\mathbf{r}_{cm} = \frac{\sum_{i=1}^{N} m_i \, \mathbf{r}_i}{\sum_{i=1}^{N} m_i}. \tag{6.20}$$

According to Eq. (6.19), the motion of the centre of mass is equivalent to that which would be obtained if all the mass contained in the system were collected at the centre of mass, and this conglomerate mass were then acted upon by the net external force. As before, the motion of the centre of mass is likely to be far simpler than the motions of the component masses.

Suppose that there is zero net external force acting on the system, so that $\mathbf{F} = \mathbf{0}$. In this case, Eq. (6.19) implies that the centre of mass moves with uniform velocity in a straight-line. In other words, the velocity of the centre of mass,

$$\dot{\mathbf{r}}_{cm} = \frac{\sum_{i=1}^{N} m_i \, \dot{\mathbf{r}}_i}{\sum_{i=1}^{N} m_i}, \tag{6.21}$$

is a constant of the motion. Now, the momentum of the ith object takes the form $\mathbf{p}_i = m_i \, \dot{\mathbf{r}}_i$. Hence, the total momentum of the system is written

$$\mathbf{P} = \sum_{i=1}^{N} m_i \, \dot{\mathbf{r}}_i. \tag{6.22}$$

A comparison of Eqs. (6.21) and (6.22) suggests that $\mathbf{P}$ is also a constant of the motion when zero net external force acts on the system. Finally, Eq. (6.19) can be rewritten

$$\frac{d\mathbf{P}}{dt} = \mathbf{F}. \tag{6.23}$$

In other words, the time derivative of the total momentum is equal to the net external force acting on the system.

It is clear, from the above discussion, that most of the important results obtained in the previous section, for the case of a two-component system moving in 1-dimension, also apply to a multi-component system moving in 3-dimensions.

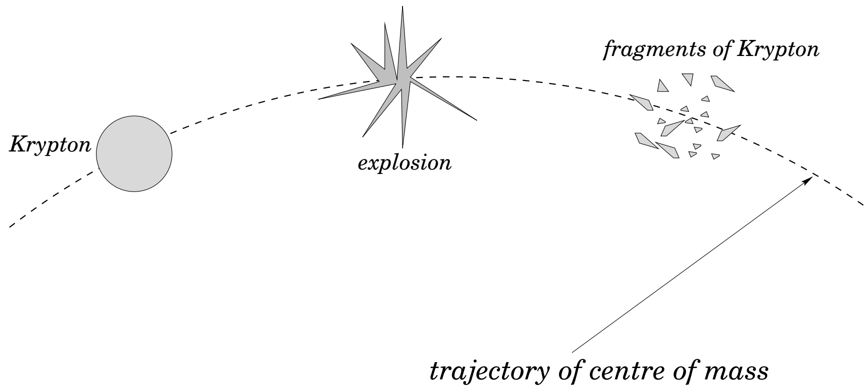


Figure 49: *The unfortunate history of the planet Krypton.*

As an illustration of the points raised in the above discussion, let us consider the unfortunate history of the planet Krypton. As you probably all know, Krypton—Superman's home planet—eventually exploded. Note, however, that before, during, and after this explosion the net external force acting on Krypton, or the fragments of Krypton—namely, the gravitational attraction to Krypton's sun—remained the same. In other words, the forces responsible for the explosion can be thought of as large, transitory, *internal* forces. We conclude that the motion of the centre of mass of Krypton, or the fragments of Krypton, was *unaffected* by the explosion. This follows, from Eq. (6.19), since the motion of the centre of mass is independent of internal forces. Before the explosion, the planet Krypton presumably executed a standard elliptical orbit around Krypton's sun. We conclude that, after the explosion, the fragments of Krypton (or, to be more exact, the centre of mass of these fragments) continued to execute *exactly the same* orbit. See Fig. 49.

6.4 Rocket science

A rocket engine is the only type of propulsion device that operates effectively in outer space. As shown in Fig. 50, a rocket works by ejecting a propellant at high velocity from its rear end. The rocket exerts a backward force on the propellant, in order to eject it, and, by Newton's third law, the propellant exerts an equal and opposite force on the rocket, which propels it forward.

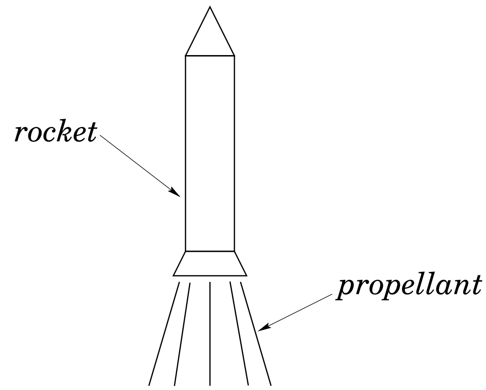


Figure 50: *A rocket.*

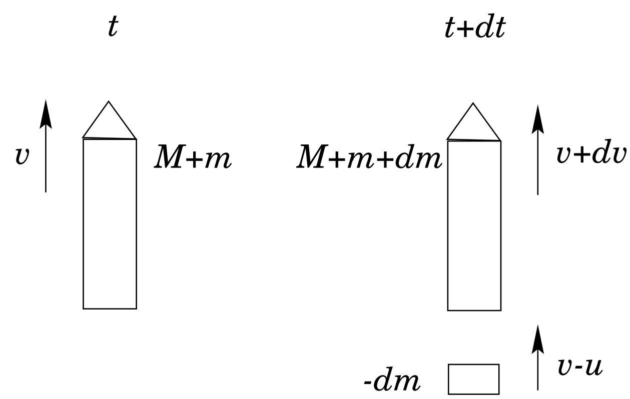


Figure 51: *Derivation of the rocket equation.*

Let us attempt to find the equation of motion of a rocket. Let M be the fixed mass of the rocket engine and the payload, and $m(t)$ the total mass of the propellant contained in the rocket's fuel tanks at time t. Suppose that the rocket engine ejects the propellant at some fixed velocity u *relative to the rocket*. Let us examine the rocket at two closely spaced instances in time. Suppose that at time t the rocket and propellant, whose total mass is $M + m$, are traveling with instantaneous velocity v. Suppose, further, that between times t and $t + dt$ the rocket ejects a quantity of propellant of mass $-dm$ (*n.b.*, dm is understood to be negative, so this represents a positive mass) which travels with velocity $v - u$ (*i.e.*, velocity $-u$ in the instantaneous rest frame of the rocket). As a result of the fuel ejection, the velocity of the rocket at time $t + dt$ is boosted to $v + dv$, and its total mass becomes $M + m + dm$. See Fig. 51.

Now, there is zero external force acting on the system, since the rocket is assumed to be in outer space. It follows that the total momentum of the system is a constant of the motion. Hence, we can equate the momenta evaluated at times t and $t + dt$:

$$(M + m)\, v = (M + m + dm)\, (v + dv) + (-dm)\, (v - u). \tag{6.24}$$

Neglecting second order quantities (*i.e.*, $dm\, dv$), the above expression yields

$$0 = (M + m)\, dv + u\, dm. \tag{6.25}$$

Rearranging, we obtain

$$\frac{dv}{u} = -\frac{dm}{M + m}. \tag{6.26}$$

Let us integrate the above equation between an initial time at which the rocket is fully fueled—*i.e.*, $m = m_p$, where m_p is the maximum mass of propellant that the rocket can carry—but stationary, and a final time at which the mass of the fuel is m and the velocity of the rocket is v. Hence,

$$\int_0^v \frac{dv}{u} = -\int_{m_p}^{m} \frac{dm}{M + m}. \tag{6.27}$$

It follows that

$$\left[\frac{v}{u}\right]_{v=0}^{v=v} = -\left[\ln(M + m)\right]_{m=m_p}^{m=m}, \tag{6.28}$$

which yields

$$v = u\, \ln\left(\frac{M + m_p}{M + m}\right). \tag{6.29}$$

The final velocity of the rocket (*i.e.*, the velocity attained by the time the rocket has exhausted its fuel, so that $m = 0$) is

$$v_f = u\, \ln\left(1 + \frac{m_p}{M}\right). \tag{6.30}$$

Note that, unless the initial mass of the fuel exceeds the fixed mass of the rocket by many orders of magnitude (which is highly unlikely), the final velocity v_f of the rocket is similar to the velocity u with which fuel is ejected from the rear of the rocket in its instantaneous rest frame. This follows because $\ln x \sim O(1)$, unless x becomes extremely large.

Let us now consider the factors which might influence the design of a rocket for use in interplanetary or interstellar travel. Since the distances involved in such travel are vast, it is important that the rocket's final velocity be made as large as possible, otherwise the journey is going to take an unacceptably long time. However, as we have just seen, the factor which essentially determines the final velocity v_f of a rocket is the speed of ejection u of the propellant relative to the rocket. Broadly speaking, v_f can never significantly exceed u. It follows that a rocket suitable for interplanetary or interstellar travel should have as high an ejection speed as practically possible. Now, ordinary chemical rockets (the kind which powered the Apollo moon program) can develop enormous thrusts, but are limited to ejection velocities below about $5000\,\text{m/s}$. Such rockets are ideal for lifting payloads out of the Earth's gravitational field, but their relatively low ejection velocities render them unsuitable for long distance space travel. A new type of rocket engine, called an *ion thruster*, is currently under development: ion thrusters operate by accelerating ions electrostatically to great velocities, and then ejecting them. Although ion thrusters only generate very small thrusts, compared to chemical rockets, their much larger ejection velocities (up to 100 times those of chemical rockets) makes them far more suitable for interplanetary or interstellar space travel. The first spacecraft to employ an ion thruster was the Deep Space 1 probe, which was launched from Cape Canaveral on October 24, 1998: this probe successfully encountered the asteroid 9969 Braille in July, 1999.

6.5 Impulses

Suppose that a ball of mass m and *speed* u_i strikes an immovable wall normally and rebounds with *speed* u_f. See Fig. 52. Clearly, the momentum of the ball is changed by the collision with the wall, since the direction of the ball's velocity is reversed. It follows that the wall must exert a force on the ball, since force is the rate of change of momentum. This force is generally very large, but is only exerted for the short instance in time during which the ball is in physical contact with the wall. As we have already mentioned, physicists generally refer to such a force as an *impulsive* force.

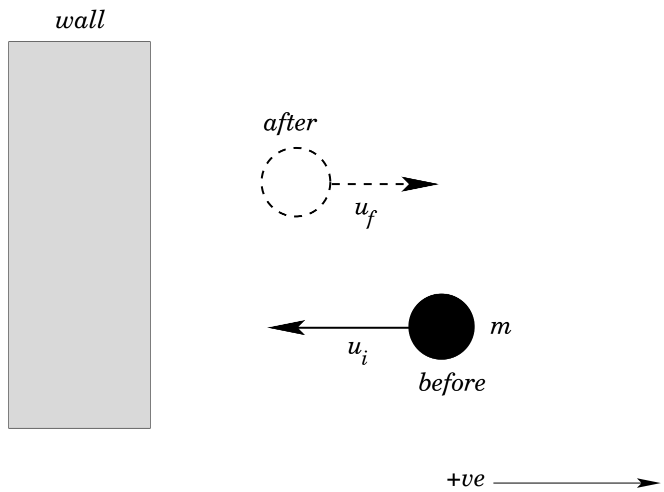


Figure 52: *A ball bouncing off a wall.*

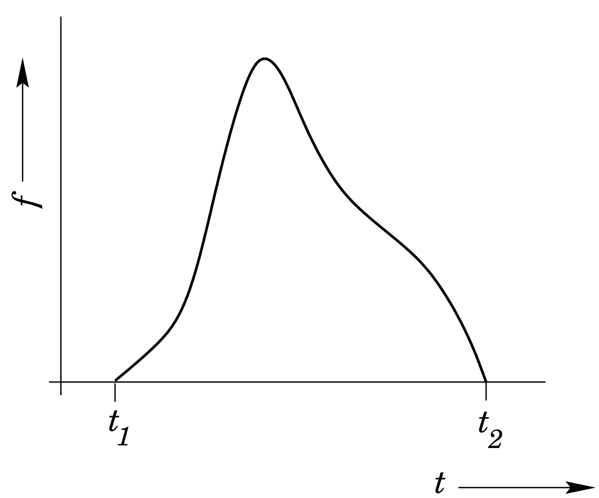


Figure 53: *An impulsive force.*

Figure 53 shows the typical time history of an impulsive force, $f(t)$. It can be seen that the force is only non-zero in the short time interval t_1 to t_2. It is helpful to define a quantity known as the net *impulse*, I, associated with $f(t)$:

$$I = \int_{t_1}^{t_2} f(t)\,dt. \tag{6.31}$$

In other words, I is the total area under the $f(t)$ curve shown in Fig. 53.

Consider a object subject to the impulsive force pictured in Fig. 53. Newton's second law of motion yields

$$\frac{dp}{dt} = f, \tag{6.32}$$

where p is the momentum of the object. Integrating the above equation, making use of the definition (6.31), we obtain

$$\Delta p = I. \tag{6.33}$$

Here, $\Delta p = p_f - p_i$, where p_i is the momentum before the impulse, and p_f is the momentum after the impulse. We conclude that the net change in momentum of an object subject to an impulsive force is equal to the total impulse associated with that force. For instance, the net change in momentum of the ball bouncing off the wall in Fig. 52 is $\Delta p = m\,u_f - m\,(-u_i) = m\,(u_f + u_i)$. [Note: The initial *velocity* is $-u_i$, since the ball is initially moving in the negative direction.] It follows that the net impulse imparted to the ball by the wall is $I = m\,(u_f + u_i)$. Suppose that we know the ball was only in physical contact with the wall for the short time interval Δt. We conclude that the *average force* $\bar{f}$ exerted on the ball during this time interval was

$$\bar{f} = \frac{I}{\Delta t}. \tag{6.34}$$

The above discussion is only relevant to 1-dimensional motion. However, the generalization to 3-dimensional motion is fairly straightforward. Consider an impulsive force $\mathbf{f}(t)$, which is only non-zero in the short time interval t_1 to t_2. The vector impulse associated with this force is simply

$$\mathbf{I} = \int_{t_1}^{t_2} \mathbf{f}(t)\,dt. \tag{6.35}$$

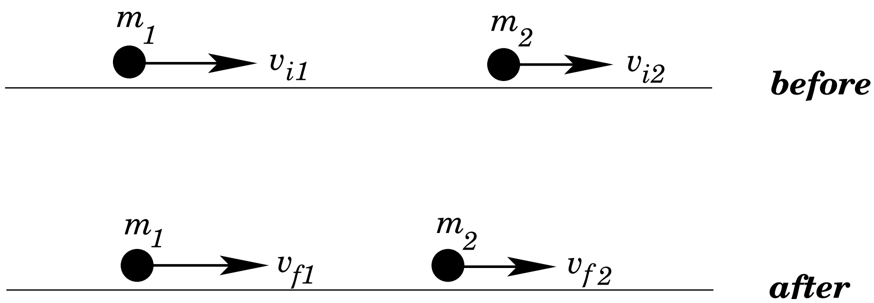


Figure 54: *A collision in 1-dimension.*

The net change in momentum of an object subject to $\mathbf{f}(t)$ is

$$\Delta\mathbf{p} = \mathbf{I}. \tag{6.36}$$

Finally, if $t_2 - t_1 = \Delta t$, then the average force experienced by the object in the time interval t_1 to t_2 is

$$\bar{\mathbf{f}} = \frac{\mathbf{I}}{\Delta t}. \tag{6.37}$$

6.6 Collisions in 1-dimension

Consider two objects of mass m_1 and m_2, respectively, which are free to move in 1-dimension. Suppose that these two objects collide. Suppose, further, that both objects are subject to zero net force when they are not in contact with one another. This situation is illustrated in Fig. 54.

Both before and after the collision, the two objects move with *constant velocity*. Let v_{i1} and v_{i2} be the velocities of the first and second objects, respectively, before the collision. Likewise, let v_{f1} and v_{f2} be the velocities of the first and second objects, respectively, after the collision. During the collision itself, the first object exerts a large transitory force f_{21} on the second, whereas the second object exerts an equal and opposite force $f_{12} = -f_{21}$ on the first. In fact, we can model the collision as equal and opposite *impulses* given to the two objects at the instant in time when they come together.

We are clearly considering a system in which there is zero net external force (the forces associated with the collision are internal in nature). Hence, the total momentum of the system is a conserved quantity. Equating the total momenta before and after the collision, we obtain

$$m_1 v_{i1} + m_2 v_{i2} = m_1 v_{f1} + m_2 v_{f2}. \tag{6.38}$$

This equation is valid for *any* 1-dimensional collision, irrespective its nature. Note that, assuming we know the masses of the colliding objects, the above equation only fully describes the collision if we are given the initial velocities of both objects, and the final velocity of at least one of the objects. (Alternatively, we could be given both final velocities and only one of the initial velocities.)

There are many different types of collision. An *elastic* collision is one in which the total kinetic energy of the two colliding objects is the same before and after the collision. Thus, for an elastic collision we can write

$$\frac{1}{2}\, m_1 v_{i1}^2 + \frac{1}{2}\, m_2 v_{i2}^2 = \frac{1}{2}\, m_1 v_{f1}^2 + \frac{1}{2}\, m_2 v_{f2}^2, \tag{6.39}$$

in addition to Eq. (6.38). Hence, in this case, the collision is fully specified once we are given the two initial velocities of the colliding objects. (Alternatively, we could be given the two final velocities.)

The majority of collisions occurring in real life are not elastic in nature. Some fraction of the initial kinetic energy of the colliding objects is usually converted into some other form of energy—generally heat energy, or energy associated with the mechanical deformation of the objects—during the collision. Such collisions are termed *inelastic*. For instance, a large fraction of the initial kinetic energy of a typical automobile accident is converted into mechanical energy of deformation of the two vehicles. Inelastic collisions also occur during squash/racquetball/handball games: in each case, the ball becomes warm to the touch after a long game, because some fraction of the ball's kinetic energy of collision with the walls of the court has been converted into heat energy. Equation (6.38) remains valid for inelastic collisions—however, Eq. (6.39) is invalid. Thus, generally speaking, an inelastic collision is only fully characterized when we are given the initial velocities of both objects, and the final velocity of at least one of the objects. There is, however, a special case of an inelastic collision—called a *totally inelastic* collision—which is fully characterized once we are given the initial velocities of the colliding objects. In a totally inelastic collision, the two objects *stick together* after the collision, so that $v_{f1} = v_{f2}$.

Let us, now, consider elastic collisions in more detail. Suppose that we transform to a frame of reference which co-moves with the centre of mass of the system. The motion of a multi-component system often looks particularly simple when viewed in such a frame. Since the system is subject to zero net external force, the velocity of the centre of mass is *invariant*, and is given by

$$v_{cm} = \frac{m_1\, v_{i1} + m_2\, v_{i2}}{m_1 + m_2} = \frac{m_1\, v_{f1} + m_2\, v_{f2}}{m_1 + m_2}. \tag{6.40}$$

An object which possesses a velocity v in our original frame of reference—henceforth, termed the *laboratory frame*—possesses a velocity $v' = v - v_{cm}$ in the centre of mass frame. It is easily demonstrated that

$$v'_{i1} = -\frac{m_2}{m_1 + m_2}\,(v_{i2} - v_{i1}), \tag{6.41}$$

$$v'_{i2} = +\frac{m_1}{m_1 + m_2}\,(v_{i2} - v_{i1}), \tag{6.42}$$

$$v'_{f1} = -\frac{m_2}{m_1 + m_2}\,(v_{f2} - v_{f1}), \tag{6.43}$$

$$v'_{f2} = +\frac{m_1}{m_1 + m_2}\,(v_{f2} - v_{f1}). \tag{6.44}$$

The above equations yield

$$-\,p'_{i1} = p'_{i2} = \mu\,(v_{i2} - v_{i1}), \tag{6.45}$$

$$-p'_{f1} = p'_{f2} = \mu\,(v_{f2} - v_{f1}), \tag{6.46}$$

where $\mu = m_1\, m_2/(m_1 + m_2)$ is the so-called *reduced mass*, and $p'_{i1} = m_1\, v'_{i1}$ is the initial momentum of the first object in the centre of mass frame, *etc*. In other words, when viewed in the centre of mass frame, the two objects approach one another with *equal and opposite momenta* before the collision, and diverge from one another with equal and opposite momenta after the collision. Thus, the centre of mass momentum conservation equation,

$$p'_{i1} + p'_{i2} = p'_{f1} + p'_{f2}, \tag{6.47}$$

is trivially satisfied, because both the left- and right-hand sides are zero. Incidentally, this result is valid for both elastic *and* inelastic collisions.

The centre of mass kinetic energy conservation equation takes the form

$$\frac{p_{i1}'^{\,2}}{2\,m_1}+\frac{p_{i2}'^{\,2}}{2\,m_2}=\frac{p_{f1}'^{\,2}}{2\,m_1}+\frac{p_{f2}'^{\,2}}{2\,m_2}. \tag{6.48}$$

Note, incidentally, that if energy and momentum are conserved in the laboratory frame then they must also be conserved in the centre of mass frame. A comparison of Eqs. (6.45), (6.46), and (6.48) yields

$$(v_{i2}-v_{i1})=-(v_{f2}-v_{f1}). \tag{6.49}$$

In other words, the *relative velocities* of the colliding objects are *equal and opposite* before and after the collision. This is true in *all* frames of reference, since relative velocities are frame invariant. Note, however, that this result *only* applies to fully elastic collisions.

Equations (6.38) and (6.49) can be combined to give the following pair of equations which fully specify the final velocities (in the laboratory frame) of two objects which collide elastically, given their initial velocities:

$$v_{f1} = \frac{(m_1-m_2)}{m_1+m_2}\,v_{i1}+\frac{2\,m_2}{m_1+m_2}\,v_{i2}, \tag{6.50}$$

$$v_{f2} = \frac{2\,m_1}{m_1+m_2}\,v_{i1}-\frac{(m_1-m_2)}{m_1+m_2}\,v_{i2}. \tag{6.51}$$

Let us, now, consider some special cases. Suppose that two *equal mass* objects collide elastically. If $m_1 = m_2$ then Eqs. (6.50) and (6.51) yield

$$v_{f1} = v_{i2}, \tag{6.52}$$

$$v_{f2} = v_{i1}. \tag{6.53}$$

In other words, the two objects simply *exchange velocities* when they collide. For instance, if the second object is stationary and the first object strikes it head-on with velocity v then the first object is brought to a halt whereas the second object moves off with velocity v. It is possible to reproduce this effect in pool by striking the cue ball with great force in such a manner that it *slides*, rather that rolls, over the table—in this case, when the cue ball strikes another ball head-on it comes to a *complete halt*, and the other ball is propelled forward very rapidly. Incidentally, it is necessary to prevent the cue ball from rolling, because rolling motion is not taken into account in our analysis, and actually changes the answer.

Suppose that the second object is much more massive than the first (*i.e.*, $m_2 \gg m_1$) and is initially at rest (*i.e.*, $v_{i2} = 0$). In this case, Eqs. (6.50) and (6.51) yield

$$v_{f1} \simeq -v_{i1}, \tag{6.54}$$

$$v_{f2} \simeq 0. \tag{6.55}$$

In other words, the velocity of the light object is effectively *reversed* during the collision, whereas the massive object remains approximately at rest. Indeed, this is the sort of behaviour we expect when an object collides elastically with an immovable obstacle: *e.g.*, when an elastic ball bounces off a brick wall.

Suppose, finally, that the second object is much lighter than the first (*i.e.*, $m_2 \ll m_1$) and is initially at rest (*i.e.*, $v_{i2} = 0$). In this case, Eqs. (6.50) and (6.51) yield

$$v_{f1} \simeq v_{i1}, \tag{6.56}$$

$$v_{f2} \simeq 2\,v_{i1}. \tag{6.57}$$

In other words, the motion of the massive object is essentially unaffected by the collision, whereas the light object ends up going *twice* as fast as the massive one.

Let us, now, consider totally inelastic collisions in more detail. In a totally inelastic collision the two objects stick together after colliding, so they end up moving with the same final velocity $v_f = v_{f1} = v_{f2}$. In this case, Eq. (6.38) reduces to

$$v_f = \frac{m_1\,v_{i1} + m_2\,v_{i2}}{m_1 + m_2} = v_{cm}. \tag{6.58}$$

In other words, the common final velocity of the two objects is equal to the centre of mass velocity of the system. This is hardly a surprising result. We have already seen that in the centre of mass frame the two objects must diverge with *equal and opposite momenta* after the collision. However, in a totally inelastic collision these

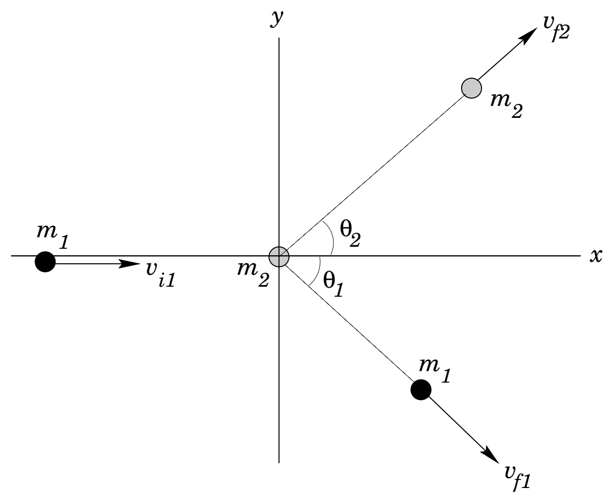


Figure 55: *A collision in 2-dimensions.*

two momenta must also be *equal* (since the two objects stick together). The only way in which this is possible is if the two objects remain *stationary* in the centre of mass frame after the collision. Hence, the two objects move with the centre of mass velocity in the laboratory frame.

Suppose that the second object is initially at rest (*i.e.*, $v_{i2} = 0$). In this special case, the common final velocity of the two objects is

$$v_f = \frac{m_1}{m_1 + m_2}\, v_{i1}. \tag{6.59}$$

Note that the first object is slowed down by the collision. The fractional loss in kinetic energy of the system due to the collision is given by

$$f = \frac{K_i - K_f}{K_i} = \frac{m_1\, v_{i1}^2 - (m_1 + m_2)\, v_f^2}{m_1\, v_{i1}^2} = \frac{m_2}{m_1 + m_2}. \tag{6.60}$$

The loss in kinetic energy is small if the (initially) stationary object is much lighter than the moving object (*i.e.*, if $m_2 \ll m_1$), and almost 100% if the moving object is much lighter than the stationary one (*i.e.*, if $m_2 \gg m_1$). Of course, the lost kinetic energy of the system is converted into some other form of energy: *e.g.*, heat energy.

6.7 Collisions in 2-dimensions

Suppose that an object of mass m_1, moving with initial speed v_{i1}, strikes a second object, of mass m_2, which is initially at rest. Suppose, further, that the collision is not head-on, so that after the collision the first object moves off at an angle θ_1 to its initial direction of motion, whereas the second object moves off at an angle θ_2 to this direction. Let the final speeds of the two objects be v_{f1} and v_{f2}, respectively. See Fig. 55.

We are again considering a system in which there is zero net external force (the forces associated with the collision are internal in nature). It follows that the total momentum of the system is a conserved quantity. However, unlike before, we must now treat the total momentum as a *vector* quantity, since we are no longer dealing with 1-dimensional motion. Note that if the collision takes place wholly within the x-y plane, as indicated in Fig. 55, then it is sufficient to equate the x- and y- components of the total momentum before and after the collision.

Consider the x-component of the system's total momentum. Before the collision, the total x-momentum is simply $m_1\,v_{i1}$, since the second object is initially stationary, and the first object is initially moving along the x-axis with speed v_{i1}. After the collision, the x-momentum of the first object is $m_1\,v_{f1}\,\cos\theta_1$: *i.e.*, m_1 times the x-component of the first object's final velocity. Likewise, the final x-momentum of the second object is $m_2\,v_{f2}\,\cos\theta_2$. Hence, momentum conservation in the x-direction yields

$$m_1\,v_{i1} = m_1\,v_{f1}\,\cos\theta_1 + m_2\,v_{f2}\,\cos\theta_2. \tag{6.61}$$

Consider the y-component of the system's total momentum. Before the collision, the total y-momentum is zero, since there is initially no motion along the y-axis. After the collision, the y-momentum of the first object is $-m_1\,v_{f1}\,\sin\theta_1$: *i.e.*, m_1 times the y-component of the first object's final velocity. Likewise, the final y-momentum of the second object is $m_2\,v_{f2}\,\sin\theta_2$. Hence, momentum conservation in the y-direction yields

$$m_1\,v_{f1}\,\sin\theta_1 = m_2\,v_{f2}\,\sin\theta_2. \tag{6.62}$$

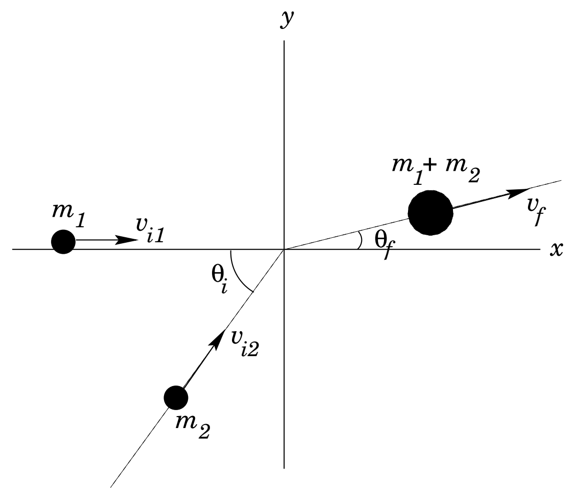


Figure 56: *A totally inelastic collision in 2-dimensions.*

For the special case of an *elastic* collision, we can equate the total kinetic energies of the two objects before and after the collision. Hence, we obtain

$$\frac{1}{2}\, m_1\, v_{i1}^2 = \frac{1}{2}\, m_1\, v_{f1}^2 + \frac{1}{2}\, m_2\, v_{f2}^2. \tag{6.63}$$

Given the initial conditions (*i.e.*, m_1, m_2, and v_{i1}), we have a system of *three* equations [*i.e.*, Eqs. (6.61), (6.62), and (6.63)] and *four* unknowns (*i.e.*, θ_1, θ_2, v_{f1}, and v_{f2}). Clearly, we cannot uniquely solve such a system without being given additional information: *e.g.*, the direction of motion or speed of one of the objects after the collision.

Figure 56 shows a 2-dimensional totally inelastic collision. In this case, the first object, mass m_1, initially moves along the x-axis with speed v_{i1}. On the other hand, the second object, mass m_2, initially moves at an angle θ_i to the x-axis with speed v_{i2}. After the collision, the two objects stick together and move off at an angle θ_f to the x-axis with speed v_f. Momentum conservation along the x-axis yields

$$m_1\, v_{i1} + m_2\, v_{i2}\, \cos\theta_i = (m_1 + m_2)\, v_f\, \cos\theta_f. \tag{6.64}$$

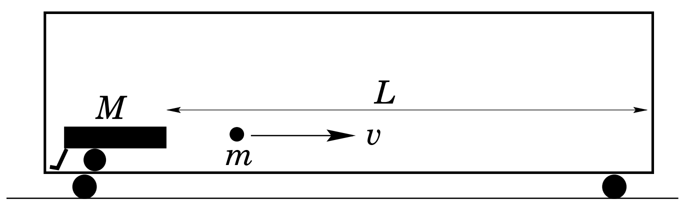
Likewise, momentum conservation along the y-axis gives

$$m_2\, v_{i2}\, \sin\theta_i = (m_1 + m_2)\, v_f\, \sin\theta_f. \tag{6.65}$$

Given the initial conditions (*i.e.*, m_1, m_2, v_{i1}, v_{i2}, and θ_i), we have a system of *two* equations [*i.e.*, Eqs. (6.64) and (6.65)] and *two* unknowns (*i.e.*, v_f and θ_f). Clearly, we should be able to find a unique solution for such a system.

Worked example 6.1: Cannon in a railway carriage

Question: A cannon is bolted to the floor of a railway carriage, which is free to move without friction along a straight track. The combined mass of the cannon and the carriage is $M = 1200\,\text{kg}$. The cannon fires a cannonball, of mass $m = 1.2\,\text{kg}$, horizontally with velocity $v = 115\,\text{m/s}$. The cannonball travels the length of the carriage, a distance $L = 85\,\text{m}$, and then becomes embedded in the carriage's end wall. What is the recoil speed of the carriage right after the cannon is fired? What is the velocity of the carriage after the cannonball strikes the far wall? What net distance, and in what direction, does the carriage move as a result of the firing of the cannon?



Answer: Conservation of momentum implies that the net horizontal momentum of the system is the same before and after the cannon is fired. The momentum before the cannon is fired is zero, since nothing is initially moving. Hence, we can also set the momentum after the cannon is fired to zero, giving

$$0 = M\,u + m\,v,$$

where u is the recoil velocity of the carriage. It follows that

$$u = -\frac{m}{M}\,v = -\frac{1.2 \times 115}{1200} = -0.115\,\text{m/s}.$$

The minus sign indicates that the recoil velocity of the carriage is in the opposite direction to the direction of motion of the cannonball. Hence, the recoil *speed* of the carriage is $|u| = 0.115\,\text{m/s}$.

Suppose that, after the cannonball strikes the far wall of the carriage, both the cannonball and the carriage move with common velocity w. Conservation of momentum implies that the net horizontal momentum of the system is the same before and after the collision. Hence, we can write

$$M\,u + m\,v = (M + m)\,w.$$

However, we have already seen that $M\,u + m\,v = 0$. It follows that $w = 0$: in other words, the carriage is brought to a complete halt when the cannonball strikes its far wall.

In the frame of reference of the carriage, the cannonball moves with velocity $v - u$ after the cannon is fired. Hence, the time of flight of the cannonball is

$$t = \frac{L}{v - u} = \frac{85}{115 + 0.115} = 0.738\,\text{s}.$$

The distance moved by the carriage in this time interval is

$$d = u\,t = -0.115 \times 0.738 = -0.0849\,\text{m}.$$

Thus, the carriage moves 8.49 cm in the opposite direction to the direction of motion of the cannonball.

Worked example 6.2: Hitting a softball

Question: A softball of mass $m = 0.35\,\text{kg}$ is pitched at a speed of $u = 12\,\text{m/s}$. The batter hits the ball directly back to the pitcher at a speed of $v = 21\,\text{m/s}$. The bat acts on the ball for $t = 0.01\,\text{s}$. What impulse is imparted by the bat to the ball? What average force is exerted by the bat on the ball?

Answer: The initial momentum of the softball is $-m\,u$, whereas its final momentum is $m\,v$. Here, the final direction of motion of the softball is taken to be positive. Thus, the net change in momentum of the softball due to its collision with the bat is

$$\Delta p = m\,v - (-)\,m\,u = 0.35 \times (21 + 12) = 11.55\,\text{N\,s}.$$

By definition, the net momentum change is equal to the impulse imparted by the bat, so

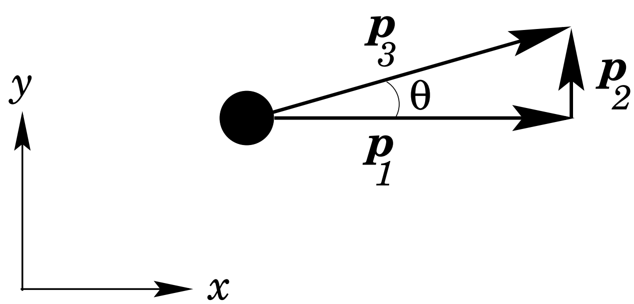
$$I = \Delta p = 11.55\,\mathrm{N\,s}.$$

The average force exerted by the bat on the ball is simply the net impulse divided by the time interval over which the ball is in contact with the bat. Hence,

$$\bar{f} = \frac{I}{t} = \frac{11.55}{0.01} = 1155.0\,\mathrm{N}.$$

Worked example 6.3: Skater and medicine ball

Question: A skater of mass $M = 120\,\mathrm{kg}$ is skating across a pond with uniform velocity $v = 8\,\mathrm{m/s}$. One of the skater's friends, who is standing at the edge of the pond, throws a medicine ball of mass $m = 20\,\mathrm{kg}$ with velocity $u = 3\,\mathrm{m/s}$ to the skater, who catches it. The direction of motion of the ball is perpendicular to the initial direction of motion of the skater. What is the final speed of the skater? What is the final direction of motion of the skater relative to his/her initial direction of motion? Assume that the skater moves without friction.



Answer: Suppose that the skater is initially moving along the x-axis, whereas the initial direction of motion of the medicine ball is along the y-axis. The skater's initial momentum is

$$\mathbf{p}_1 = (M\,v,\,0) = (120 \times 8,\,0) = (960,\,0)\,\mathrm{N\,s}.$$

Likewise, the initial momentum of the medicine ball is

$$\mathbf{p}_2 = (0,\,m\,u) = (0,\,20 \times 3) = (0,\,60)\,\mathrm{N\,s}.$$

After the skater catches the ball, the combined momentum of the skater and the ball is

$$\mathbf{p}_3 = \mathbf{p}_1 + \mathbf{p}_2 = (960,\ 60)\,\mathrm{N\,s}.$$

This follows from momentum conservation. The final speed of the skater (and the ball) is

$$v' = \frac{|\mathbf{p}_3|}{M + m} = \frac{\sqrt{960^2 + 60^2}}{120 + 20} = 6.87\,\mathrm{m/s}.$$

The final direction of motion of the skater is parameterized by the angle θ (see the above diagram), where

$$\theta = \tan^{-1}\left(\frac{|\mathbf{p}_2|}{|\mathbf{p}_1|}\right) = \tan^{-1}\left(\frac{60}{960}\right) = 3.58^\circ.$$

Worked example 6.4: Bullet and block

Question: A bullet of mass $m = 12\,\mathrm{g}$ strikes a stationary wooden block of mass $M = 5.2\,\mathrm{kg}$ standing on a frictionless surface. The block, with the bullet embedded in it, acquires a velocity of $v = 1.7\,\mathrm{m/s}$. What was the velocity of the bullet before it struck the block? What fraction of the bullet's initial kinetic energy is lost (*i.e.*, dissipated) due to the collision with the block?

Answer: Let u be the initial velocity of the bullet. Momentum conservation requires the total horizontal momentum of the system to be the same before and after the bullet strikes the block. The initial momentum of the system is simply $m\,u$, since the block is initially at rest. The final momentum is $(M + m)\,v$, since both the block and the bullet end up moving with velocity v. Hence,

$$m\,u = (M + m)\,v,$$

giving

$$u = \frac{M + m}{m}\,v = \frac{(0.012 + 5.2) \times 1.7}{0.012} = 738.4\,\mathrm{m/s}.$$

The initial kinetic energy of the bullet is

$$K_i = \frac{1}{2}\,m\,u^2 = 0.5 \times 0.012 \times 738.4^2 = 3.2714\,\mathrm{kJ}.$$

The final kinetic energy of the system is

$$K_f = \frac{1}{2}(M+m)\,v^2 = 0.5\times(0.012+5.2)\times 1.7^2 = 7.53\,\text{J}.$$

Hence, the fraction of the initial kinetic energy which is dissipated is

$$f = \frac{K_i - K_f}{K_i} = \frac{3.2714\times 10^3 - 7.53}{3.2714\times 10^3} = 0.9977.$$

Worked example 6.5: Elastic collision

Question: An object of mass $m_1 = 2\,\text{kg}$, moving with velocity $v_{i1} = 12\,\text{m/s}$, collides head-on with a stationary object whose mass is $m_2 = 6\,\text{kg}$. Given that the collision is elastic, what are the final velocities of the two objects. Neglect friction.

Answer: Momentum conservation yields

$$m_1\,v_{i1} = m_1\,v_{f1} + m_2\,v_{f2},$$

where v_{f1} and v_{f2} are the final velocities of the first and second objects, respectively. Since the collision is elastic, the total kinetic energy must be the same before and after the collision. Hence,

$$\frac{1}{2}\,m_1\,v_{i1}^2 = \frac{1}{2}\,m_1\,v_{f1}^2 + \frac{1}{2}\,m_2\,v_{f2}^2.$$

Let $x = v_{f1}/v_{i1}$ and $y = v_{f2}/v_{i1}$. Noting that $m_2/m_1 = 3$, the above two equations reduce to

$$1 = x + 3\,y,$$

and

$$1 = x^2 + 3\,y^2.$$

Eliminating x between the previous two expressions, we obtain

$$1 = (1-3\,y)^2 + 3\,y^2,$$

or

$$6\,y\,(2\,y-1) = 0,$$

which has the non-trivial solution $y = 1/2$. The corresponding solution for x is $x = (1 - 3\,y) = -1/2$.

It follows that the final velocity of the first object is

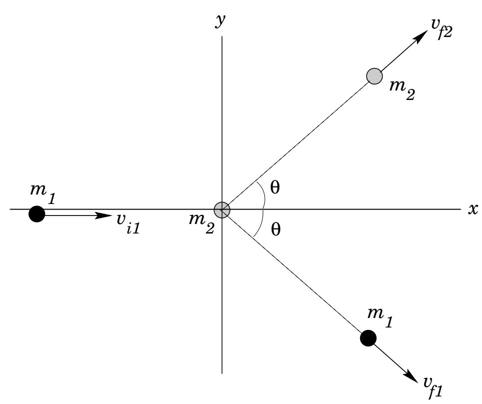
$$v_{f1} = x\,v_{i1} = -0.5 \times 12 = -6\,\text{m/s}.$$

The minus sign indicates that this object reverses direction as a result of the collision. Likewise, the final velocity of the second object is

$$v_{f2} = y\,v_{i1} = 0.5 \times 12 = 6\,\text{m/s}.$$

Worked example 6.6: 2-dimensional collision

Question: Two objects slide over a frictionless horizontal surface. The first object, mass $m_1 = 5\,\text{kg}$, is propelled with speed $v_{i1} = 4.5\,\text{m/s}$ toward the second object, mass $m_2 = 2.5\,\text{kg}$, which is initially at rest. After the collision, both objects have velocities which are directed $\theta = 30^\circ$ on either side of the original line of motion of the first object. What are the final speeds of the two objects? Is the collision elastic or inelastic?



Answer: Let us adopt the coordinate system shown in the diagram. Conservation of momentum along the x-axis yields

$$m_1\,v_{i1} = m_1\,v_{f1}\,\cos\theta + m_2\,v_{f2}\,\cos\theta.$$

Likewise, conservation of momentum along the y-axis yields

$$m_1\,v_{f1}\,\sin\theta = m_2\,v_{f2}\,\sin\theta.$$

The above pair of equations can be combined to give

$$v_{f1} = \frac{v_{i1}}{2\,\cos\theta} = \frac{4.5}{2\times\cos 30^\circ} = 2.5981\,\mathrm{m/s},$$

and

$$v_{f2} = \frac{m_1}{m_2}\,v_{f1} = \frac{5\times 2.5981}{2.5} = 5.1962\,\mathrm{m/s}.$$

The initial kinetic energy of the system is

$$K_i = \frac{1}{2}\,m_1\,v_{i1}^2 = 0.5\times 5\times 4.5^2 = 50.63\,\mathrm{J}.$$

The final kinetic energy of the system is

$$K_f = \frac{1}{2}\,m_1\,v_{f1}^2 + \frac{1}{2}\,m_2\,v_{f2}^2 = 0.5\times 5\times 2.5981^2 + 0.5\times 2.5\times 5.1962^2 = 50.63\,\mathrm{J}.$$

Since $K_i = K_f$, the collision is *elastic*.

7 Circular motion

7.1 Introduction

Up to now, we have basically only considered *rectilinear* motion: *i.e.*, motion in a straight-line. Let us now broaden our approach so as to take into account the most important type of non-rectilinear motion: namely, *circular* motion.

7.2 Uniform circular motion

Suppose that an object executes a circular orbit of radius r with *uniform* tangential speed v. The instantaneous position of the object is most conveniently specified in terms of an angle θ. See Fig. 57. For instance, we could decide that $\theta = 0^\circ$ corresponds to the object's location at $t = 0$, in which case we would write

$$\theta(t) = \omega\, t, \tag{7.1}$$

where ω is termed the *angular velocity* of the object. For a uniformly rotating object, the angular velocity is simply the angle through which the object turns in one second.

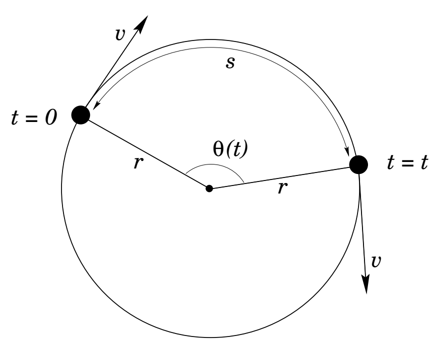


Figure 57: *Circular motion.*

Consider the motion of the object in the time interval between $t = 0$ and $t = t$. In this interval, the object rotates through an angle θ, and traces out a circular arc of length s. See Fig. 57. It is fairly obvious that the arc length s is directly proportional to the angle θ: but, what is the constant of proportionality? Well, an angle of 360° corresponds to an arc length of $2\pi r$. Hence, an angle θ must correspond to an arc length of

$$s = \frac{2\pi}{360^\circ}\, r\,\theta(^\circ). \tag{7.2}$$

At this stage, it is convenient to define a new angular unit known as a *radian* (symbol rad.). An angle measured in radians is related to an angle measured in degrees via the following simple formula:

$$\theta(\text{rad.}) = \frac{2\pi}{360^\circ}\,\theta(^\circ). \tag{7.3}$$

Thus, 360° corresponds to 2π radians, 180° corresponds to π radians, 90° corresponds to $\pi/2$ radians, and 57.296° corresponds to 1 radian. When θ is measured in radians, Eq. (7.2) simplifies greatly to give

$$s = r\,\theta. \tag{7.4}$$

Henceforth, in this course, all angles are measured in radians *by default*.

Consider the motion of the object in the short interval between times t and $t + \delta t$. In this interval, the object turns through a small angle $\delta\theta$ and traces out a short arc of length δs, where

$$\delta s = r\,\delta\theta. \tag{7.5}$$

Now $\delta s/\delta t$ (*i.e.*, distance moved per unit time) is simply the tangential velocity v, whereas $\delta\theta/\delta t$ (*i.e.*, angle turned through per unit time) is simply the angular velocity ω. Thus, dividing Eq. (7.5) by δt, we obtain

$$v = r\,\omega. \tag{7.6}$$

Note, however, that this formula is only valid if the angular velocity ω is measured in *radians per second*. From now on, in this course, all angular velocities are measured in radians per second *by default*.

An object that rotates with uniform angular velocity ω turns through ω radians in 1 second. Hence, the object turns through 2π radians (*i.e.*, it executes a complete circle) in

$$T = \frac{2\pi}{\omega} \tag{7.7}$$

seconds. Here, T is the *repetition period* of the circular motion. If the object executes a complete cycle (*i.e.*, turns through 360°) in T seconds, then the number of cycles executed per second is

$$f = \frac{1}{T} = \frac{\omega}{2\pi}. \tag{7.8}$$

Here, the *repetition frequency*, f, of the motion is measured in *cycles per second*—otherwise known as *hertz* (symbol Hz).

As an example, suppose that an object executes uniform circular motion, radius $r = 1.2\,\mathrm{m}$, at a frequency of $f = 50\,\mathrm{Hz}$ (*i.e.*, the object executes a complete rotation 50 times a second). The repetition period of this motion is simply

$$T = \frac{1}{f} = 0.02\,\mathrm{s}. \tag{7.9}$$

Furthermore, the angular frequency of the motion is given by

$$\omega = 2\pi\,f = 314.16\,\mathrm{rad./s}. \tag{7.10}$$

Finally, the tangential velocity of the object is

$$v = r\,\omega = 1.2 \times 314.16 = 376.99\,\mathrm{m/s}. \tag{7.11}$$

7.3 Centripetal acceleration

An object executing a circular orbit of radius r with uniform tangential speed v possesses a velocity vector $\mathbf{v}$ whose magnitude is constant, but whose direction is continuously changing. It follows that the object must be *accelerating*, since (vector) acceleration is the rate of change of (vector) velocity, and the (vector) velocity is indeed varying in time.

Suppose that the object moves from point P to point Q between times t and $t + \delta t$, as shown in Fig. 58. Suppose, further, that the object rotates through $\delta\theta$ radians in this time interval. The vector $\vec{PX}$, shown in the diagram, is identical to the vector $\vec{QY}$. Moreover, the angle subtended between vectors $\vec{PZ}$ and $\vec{PX}$ is

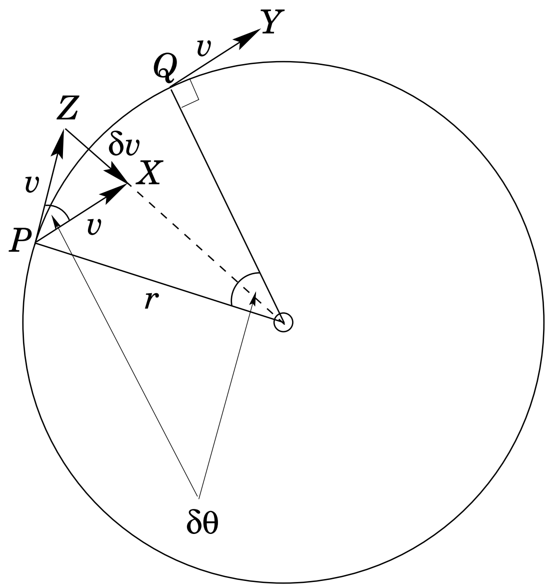


Figure 58: *Centripetal acceleration.*

simply δθ. The vector $\overrightarrow{ZX}$ represents the change in vector velocity, $\delta\mathbf{v}$, between times t and $t+\delta t$. It can be seen that this vector is directed *towards the centre of the circle*. From standard trigonometry, the length of vector $\overrightarrow{ZX}$ is

$$\delta v = 2\,v\,\sin(\delta\theta/2). \tag{7.12}$$

However, for small angles $\sin\theta \simeq \theta$, provided that θ is measured in *radians*. Hence,

$$\delta v \simeq v\,\delta\theta. \tag{7.13}$$

It follows that

$$a = \frac{\delta v}{\delta t} = v\,\frac{\delta\theta}{\delta t} = v\,\omega, \tag{7.14}$$

where $\omega = \delta\theta/\delta t$ is the angular velocity of the object, measured in *radians per second*. In summary, an object executing a circular orbit, radius r, with uniform tangential velocity v, and uniform angular velocity $\omega = v/r$, possesses an acceleration directed towards the centre of the circle—*i.e.*, a *centripetal* acceleration—of magnitude

$$a = v\,\omega = \frac{v^2}{r} = r\,\omega^2. \tag{7.15}$$

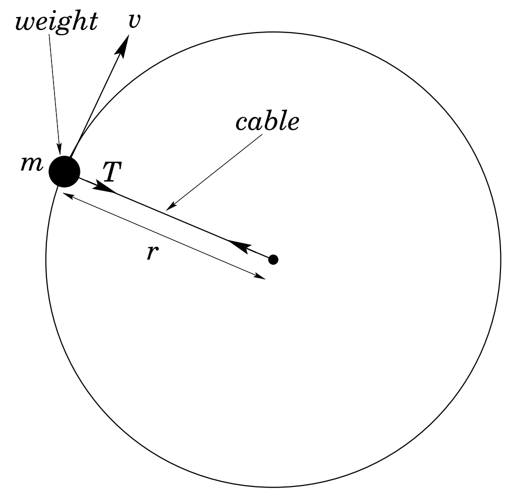


Figure 59: *Weight on the end of a cable.*

Suppose that a weight, of mass m, is attached to the end of a cable, of length r, and whirled around such that the weight executes a horizontal circle, radius r, with uniform tangential velocity v. As we have just learned, the weight is subject to a centripetal acceleration of magnitude v^2/r. Hence, the weight experiences a centripetal force

$$f = \frac{m\,v^2}{r}. \tag{7.16}$$

What provides this force? Well, in the present example, the force is provided by the *tension* T in the cable. Hence, $T = m\,v^2/r$.

Suppose that the cable is such that it snaps whenever the tension in it exceeds a certain critical value T_{max}. It follows that there is a maximum velocity with which the weight can be whirled around: namely,

$$v_{max} = \sqrt{\frac{r\,T_{max}}{m}}. \tag{7.17}$$

If v exceeds v_{max} then the cable will break. As soon as the cable snaps, the weight will cease to be subject to a centripetal force, so it will fly off—with velocity v_{max}—along the *straight-line* which is *tangential* to the circular orbit it was previously executing.

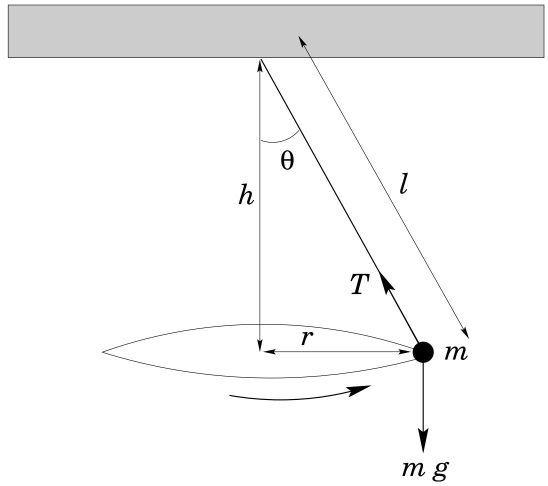


Figure 60: *A conical pendulum.*

7.4 The conical pendulum

Suppose that an object, mass m, is attached to the end of a light inextensible string whose other end is attached to a rigid beam. Suppose, further, that the object is given an initial horizontal velocity such that it executes a horizontal circular orbit of radius r with angular velocity ω. See Fig. 60. Let h be the vertical distance between the beam and the plane of the circular orbit, and let θ be the angle subtended by the string with the downward vertical.

The object is subject to two forces: the gravitational force $m\,g$ which acts vertically downwards, and the tension force T which acts upwards along the string. The tension force can be resolved into a component $T\,\cos\theta$ which acts vertically upwards, and a component $T\,\sin\theta$ which acts towards the centre of the circle. Force balance in the vertical direction yields

$$T\,\cos\theta = m\,g. \tag{7.18}$$

In other words, the vertical component of the tension force balances the weight of the object.

Since the object is executing a circular orbit, radius r, with angular velocity ω, it experiences a centripetal acceleration $\omega^2\,r$. Hence, it is subject to a centripetal force $m\,\omega^2\,r$. This force is provided by the component of the string tension which acts towards the centre of the circle. In other words,

$$T \sin\theta = m\,\omega^2\,r. \tag{7.19}$$

Taking the ratio of Eqs. (7.18) and (7.19), we obtain

$$\tan\theta = \frac{\omega^2\,r}{g}. \tag{7.20}$$

However, by simple trigonometry,

$$\tan\theta = \frac{r}{h}. \tag{7.21}$$

Hence, we find

$$\omega = \sqrt{\frac{g}{h}}. \tag{7.22}$$

Note that if l is the length of the string then $h = l\,\cos\theta$. It follows that

$$\omega = \sqrt{\frac{g}{l\,\cos\theta}}. \tag{7.23}$$

For instance, if the length of the string is $l = 0.2\,\text{m}$ and the conical angle is $\theta = 30^\circ$ then the angular velocity of rotation is given by

$$\omega = \sqrt{\frac{9.81}{0.2 \times \cos 30^\circ}} = 7.526\,\text{rad./s}. \tag{7.24}$$

This translates to a rotation frequency in cycles per second of

$$f = \frac{\omega}{2\,\pi} = 1.20\,\text{Hz}. \tag{7.25}$$

7.5 Non-uniform circular motion

Consider an object which executes *non-uniform* circular motion, as shown in Fig. 61. Suppose that the motion is confined to a 2-dimensional plane. We can specify the instantaneous position of the object in terms of its *polar coordinates* r

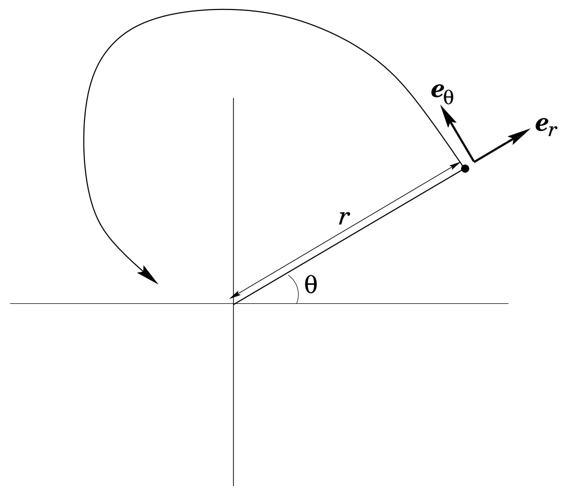


Figure 61: *Polar coordinates.*

and θ. Here, r is the radial distance of the object from the origin, whereas θ is the angular bearing of the object from the origin, measured with respect to some arbitrarily chosen direction. We imagine that both r and θ are changing in time. As an example of non-uniform circular motion, consider the motion of the Earth around the Sun. Suppose that the origin of our coordinate system corresponds to the position of the Sun. As the Earth rotates, its angular bearing θ, relative to the Sun, obviously changes in time. However, since the Earth's orbit is slightly *elliptical*, its radial distance r from the Sun also varies in time. Moreover, as the Earth moves closer to the Sun, its rate of rotation speeds up, and *vice versa*. Hence, the rate of change of θ with time is non-uniform.

Let us define two unit vectors, $\mathbf{e}_r$ and $\mathbf{e}_\theta$. Incidentally, a unit vector simply a vector whose length is unity. As shown in Fig. 61, the *radial* unit vector $\mathbf{e}_r$ always points from the origin to the instantaneous position of the object. Moreover, the *tangential* unit vector $\mathbf{e}_\theta$ is always *normal* to $\mathbf{e}_r$, in the direction of increasing θ. The position vector $\mathbf{r}$ of the object can be written

$$\mathbf{r} = r\,\mathbf{e}_r. \tag{7.26}$$

In other words, vector $\mathbf{r}$ points in the same direction as the radial unit vector $\mathbf{e}_r$, and is of length r. We can write the object's velocity in the form

$$\mathbf{v} = \dot{\mathbf{r}} = v_r\,\mathbf{e}_r + v_\theta\,\mathbf{e}_\theta, \tag{7.27}$$

whereas the acceleration is written

$$\mathbf{a} = \dot{\mathbf{v}} = a_r\,\mathbf{e}_r + a_\theta\,\mathbf{e}_\theta. \tag{7.28}$$

Here, v_r is termed the object's *radial velocity*, whilst v_θ is termed the *tangential velocity*. Likewise, a_r is the *radial acceleration*, and a_θ is the *tangential acceleration*. But, how do we express these quantities in terms of the object's polar coordinates r and θ? It turns out that this is a far from straightforward task. For instance, if we simply differentiate Eq. (7.26) with respect to time, we obtain

$$\mathbf{v} = \dot{r}\,\mathbf{e}_r + r\,\dot{\mathbf{e}}_r, \tag{7.29}$$

where $\dot{\mathbf{e}}_r$ is the time derivative of the radial unit vector—this quantity is non-zero because $\mathbf{e}_r$ *changes direction* as the object moves. Unfortunately, it is not entirely clear how to evaluate $\dot{\mathbf{e}}_r$. In the following, we outline a famous trick for calculating v_r, v_θ, *etc.* without ever having to evaluate the time derivatives of the unit vectors $\mathbf{e}_r$ and $\mathbf{e}_\theta$.

Consider a general *complex number*,

$$z = x + i\,y, \tag{7.30}$$

where x and y are real, and i is the square root of -1 (*i.e.*, $i^2 = -1$). Here, x is the real part of z, whereas y is the imaginary part. We can visualize z as a point in the so-called *complex plane*: *i.e.*, a 2-dimensional plane in which the real parts of complex numbers are plotted along one Cartesian axis, whereas the corresponding imaginary parts are plotted along the other axis. Thus, the coordinates of z in the complex plane are simply (x, y). See Fig. 62. In other words, we can use a complex number to represent a position vector in a 2-dimensional plane. Note that the length of the vector is equal to the *modulus* of the corresponding complex number. Incidentally, the modulus of $z = x + i\,y$ is defined

$$|z| = \sqrt{x^2 + y^2}. \tag{7.31}$$

Consider the complex number $e^{\,i\,\theta}$, where θ is real. A famous result in complex analysis—known as *de Moivre's theorem*—allows us to split this number into its real and imaginary components:

$$e^{\,i\,\theta} = \cos\theta + i\,\sin\theta. \tag{7.32}$$

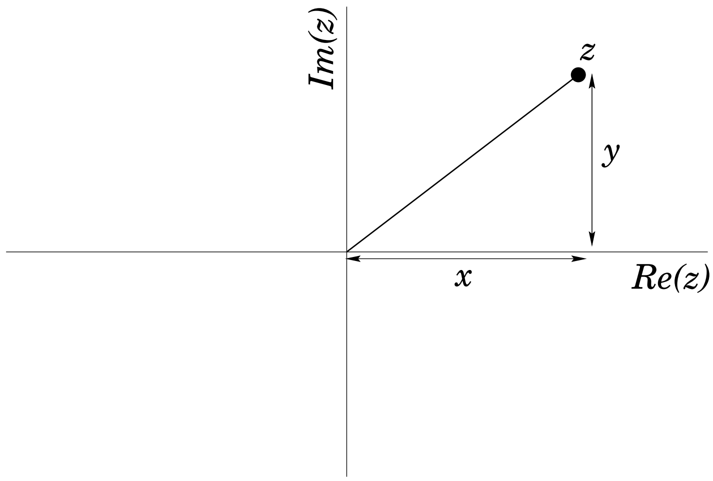


Figure 62: *Representation of a complex number in the complex plane.*

Now, as we have just discussed, we can think of $e^{i\theta}$ as representing a vector in the complex plane: the real and imaginary parts of $e^{i\theta}$ form the coordinates of the head of the vector, whereas the tail of the vector corresponds to the origin. What are the properties of this vector? Well, the length of the vector is given by

$$\left|e^{i\theta}\right| = \sqrt{\cos^2\theta + \sin^2\theta} = 1. \tag{7.33}$$

In other words, $e^{i\theta}$ represents a *unit vector*. In fact, it is clear from Fig. 63 that $e^{i\theta}$ represents the radial unit vector $\mathbf{e}_r$ for an object whose angular polar coordinate (measured anti-clockwise from the real axis) is θ. Can we also find a complex representation of the corresponding tangential unit vector $\mathbf{e}_\theta$? Actually, we can. The complex number $i\,e^{i\theta}$ can be written

$$i\,e^{i\theta} = -\sin\theta + i\,\cos\theta. \tag{7.34}$$

Here, we have just multiplied Eq. (7.32) by i, making use of the fact that $i^2 = -1$. This number again represents a unit vector, since

$$\left|i\,e^{i\theta}\right| = \sqrt{\sin^2\theta + \cos^2\theta} = 1. \tag{7.35}$$

Moreover, as is clear from Fig. 63, this vector is normal to $\mathbf{e}_r$, in the direction of increasing θ. In other words, $i\,e^{i\theta}$ represents the tangential unit vector $\mathbf{e}_\theta$.

Consider an object executing non-uniform circular motion in the complex plane. By analogy with Eq. (7.26), we can represent the instantaneous position

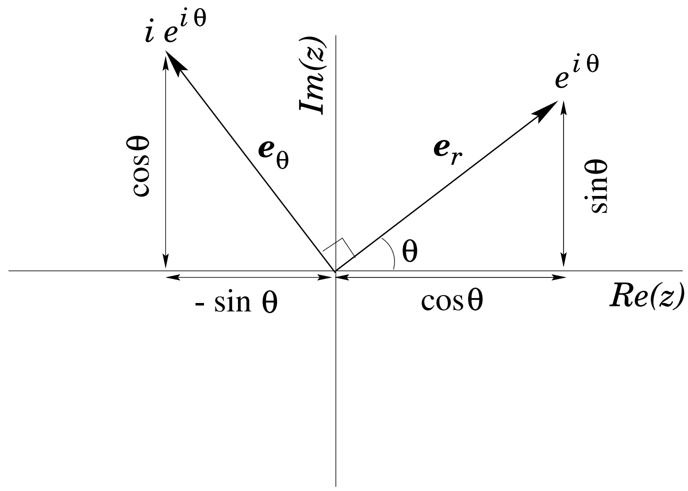


Figure 63: *Representation of the unit vectors* $\mathbf{e}_r$ *and* $\mathbf{e}_\theta$ *in the complex plane.*

vector of this object via the complex number

$$z = r\,e^{\,i\,\theta}. \tag{7.36}$$

Here, $r(t)$ is the object's radial distance from the origin, whereas $\theta(t)$ is its angular bearing relative to the real axis. Note that, in the above formula, we are using $e^{\,i\,\theta}$ to represent the radial unit vector $\mathbf{e}_r$. Now, if z represents the position vector of the object, then $\dot{z} = dz/dt$ must represent the object's velocity vector. Differentiating Eq. (7.36) with respect to time, using the standard rules of calculus, we obtain

$$\dot{z} = \dot{r}\,e^{\,i\,\theta} + r\,\dot{\theta}\,i\,e^{\,i\,\theta}. \tag{7.37}$$

Comparing with Eq. (7.27), recalling that $e^{\,i\,\theta}$ represents $\mathbf{e}_r$ and $i\,e^{\,i\,\theta}$ represents $\mathbf{e}_\theta$, we obtain

$$v_r = \dot{r}, \tag{7.38}$$

$$v_\theta = r\,\dot{\theta} = r\,\omega, \tag{7.39}$$

where $\omega = d\theta/dt$ is the object's instantaneous angular velocity. Thus, as desired, we have obtained expressions for the radial and tangential velocities of the object in terms of its polar coordinates, r and θ. We can go further. Let us differentiate $\dot{z}$ with respect to time, in order to obtain a complex number representing the object's vector acceleration. Again, using the standard rules of calculus, we obtain

$$\ddot{z} = (\ddot{r} - r\,\dot{\theta}^2)\,e^{\,i\,\theta} + (r\,\ddot{\theta} + 2\,\dot{r}\,\dot{\theta})\,i\,e^{\,i\,\theta}. \tag{7.40}$$

Comparing with Eq. (7.28), recalling that $e^{i\theta}$ represents $\mathbf{e}_r$ and $i\,e^{i\theta}$ represents $\mathbf{e}_\theta$, we obtain

$$a_r = \ddot{r} - r\,\dot{\theta}^2 = \ddot{r} - r\,\omega^2, \tag{7.41}$$

$$a_\theta = r\,\ddot{\theta} + 2\,\dot{r}\,\dot{\theta} = r\,\dot{\omega} + 2\,\dot{r}\,\omega. \tag{7.42}$$

Thus, we now have expressions for the object's radial and tangential accelerations in terms of r and θ. The beauty of this derivation is that the complex analysis has automatically taken care of the fact that the unit vectors $\mathbf{e}_r$ and $\mathbf{e}_\theta$ change direction as the object moves.

Let us now consider the commonly occurring special case in which an object executes a circular orbit at *fixed radius*, but varying angular velocity. Since the radius is fixed, it follows that $\dot{r} = \ddot{r} = 0$. According to Eqs. (7.38) and (7.39), the radial velocity of the object is zero, and the tangential velocity takes the form

$$v_\theta = r\,\omega. \tag{7.43}$$

Note that the above equation is exactly the same as Eq. (7.6)—the only difference is that we have now proved that this relation holds for non-uniform, as well as uniform, circular motion. According to Eq. (7.41), the radial acceleration is given by

$$a_r = -r\,\omega^2. \tag{7.44}$$

The minus sign indicates that this acceleration is directed towards the centre of the circle. Of course, the above equation is equivalent to Eq. (7.15)—the only difference is that we have now proved that this relation holds for non-uniform, as well as uniform, circular motion. Finally, according to Eq. (7.42), the tangential acceleration takes the form

$$a_\theta = r\,\dot{\omega}. \tag{7.45}$$

The existence of a non-zero tangential acceleration (in the former case) is the one difference between non-uniform and uniform circular motion (at constant radius).

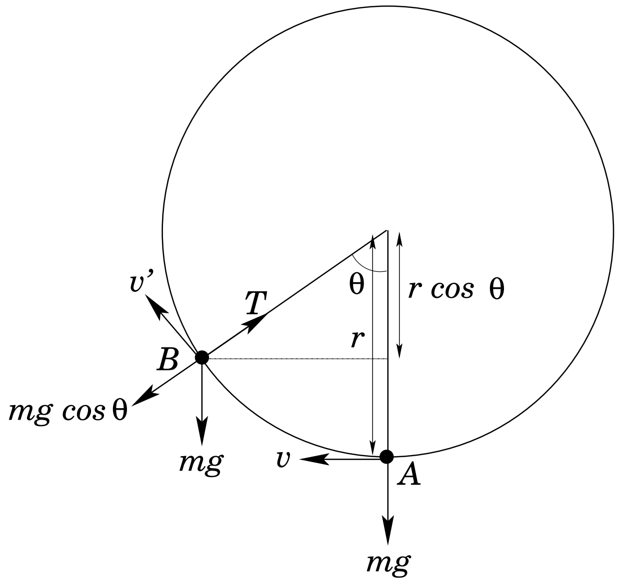


Figure 64: *Motion in a vertical circle.*

7.6 The vertical pendulum

Let us now examine an example of non-uniform circular motion. Suppose that an object of mass m is attached to the end of a light rigid rod, or light string, of length r. The other end of the rod, or string, is attached to a stationary pivot in such a manner that the object is free to execute a vertical circle about this pivot. Let θ measure the angular position of the object, measured with respect to the downward vertical. Let v be the velocity of the object at $\theta = 0^\circ$. How large do we have to make v in order for the object to execute a complete vertical circle?

Consider Fig. 64. Suppose that the object moves from point A, where its tangential velocity is v, to point B, where its tangential velocity is v'. Let us, first of all, obtain the relationship between v and v'. This is most easily achieved by considering energy conservation. At point A, the object is situated a vertical distance r below the pivot, whereas at point B the vertical distance below the pivot has been reduced to $r\cos\theta$. Hence, in moving from A to B the object gains potential energy $m\,g\,r\,(1-\cos\theta)$. This gain in potential energy must be offset by a corresponding loss in kinetic energy. Thus,

$$\frac{1}{2}\,m\,v^2 - \frac{1}{2}\,m\,v'^2 = m\,g\,r\,(1-\cos\theta), \tag{7.46}$$

which reduces to

$$v'^2 = v^2 - 2\,r\,g\,(1 - \cos\theta). \tag{7.47}$$

Let us now examine the radial acceleration of the object at point B. The radial forces acting on the object are the tension T in the rod, or string, which acts towards the centre of the circle, and the component $m\,g\,\cos\theta$ of the object's weight, which acts away from the centre of the circle. Since the object is executing circular motion with instantaneous tangential velocity v', it must experience an instantaneous acceleration v'^2/r towards the centre of the circle. Hence, Newton's second law of motion yields

$$\frac{m\,v'^2}{r} = T - m\,g\,\cos\theta. \tag{7.48}$$

Equations (7.47) and (7.48) can be combined to give

$$T = \frac{m\,v^2}{r} + m\,g\,(3\,\cos\theta - 2). \tag{7.49}$$

Suppose that the object is, in fact, attached to the end of a piece of string, rather than a rigid rod. One important property of strings is that, unlike rigid rods, they cannot support negative tensions. In other words, a string can only pull objects attached to its two ends together—it cannot push them apart. Another way of putting this is that if the tension in a string ever becomes negative then the string will become slack and collapse. Clearly, if our object is to execute a full vertical circle then then tension T in the string must remain positive for all values of θ. It is clear from Eq. (7.49) that the tension attains its minimum value when $\theta = 180^\circ$ (at which point $\cos\theta = -1$). This is hardly surprising, since $\theta = 180^\circ$ corresponds to the point at which the object attains its maximum height, and, therefore, its minimum tangential velocity. It is certainly the case that if the string tension is positive at this point then it must be positive at all other points. Now, the tension at $\theta = 180^\circ$ is given by

$$T_0 = \frac{m\,v^2}{r} - 5\,m\,g. \tag{7.50}$$

Hence, the condition for the object to execute a complete vertical circle without the string becoming slack is $T_0 > 0$, or

$$v^2 > 5\,r\,g. \tag{7.51}$$

Note that this condition is independent of the mass of the object.

Suppose that the object is attached to the end of a rigid rod, instead of a piece of string. There is now no constraint on the tension, since a rigid rod can quite easily support a negative tension (*i.e.*, it can push, as well as pull, on objects attached to its two ends). However, in order for the object to execute a complete vertical circle the square of its tangential velocity v'^2 must remain positive at all values of θ. It is clear from Eq. (7.47) that v'^2 attains its minimum value when $\theta = 180^\circ$. This is, again, hardly surprising. Thus, if v'^2 is positive at this point then it must be positive at all other points. Now, the expression for v'^2 at $\theta = 180^\circ$ is

$$(v'^2)_0 = v^2 - 4\,r\,g. \tag{7.52}$$

Hence, the condition for the object to execute a complete vertical circle is $(v'^2)_0 > 0$, or

$$v^2 > 4\,r\,g. \tag{7.53}$$

Note that this condition is slightly easier to satisfy than the condition (7.51). In other words, it is slightly easier to cause an object attached to the end of a rigid rod to execute a vertical circle than it is to cause an object attached to the end of a string to execute the same circle. The reason for this is that the rigidity of the rod helps support the object when it is situated above the pivot point.

7.7 Motion on curved surfaces

Consider a smooth rigid vertical hoop of internal radius r, as shown in Fig. 65. Suppose that an object of mass m slides without friction around the inside of this hoop. What is the motion of this object? Is it possible for the object to execute a complete vertical circle?

Suppose that the object moves from point A to point B in Fig. 65. In doing so, it gains potential energy $m\,g\,r\,(1-\cos\theta)$, where θ is the angular coordinate of the object measured with respect to the downward vertical. This gain in potential energy must be offset by a corresponding loss in kinetic energy. Thus,

$$\frac{1}{2}\,m\,v^2 - \frac{1}{2}\,m\,v'^2 = m\,g\,r\,(1-\cos\theta), \tag{7.54}$$

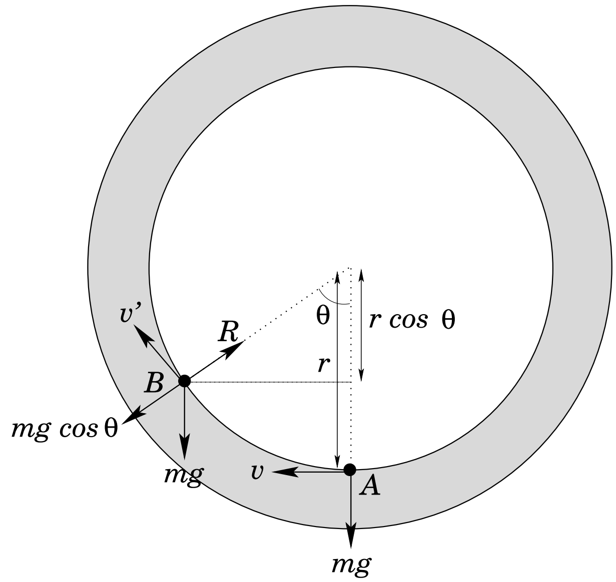


Figure 65: *Motion on the inside of a vertical hoop.*

which reduces to

$$v'^2 = v^2 - 2\,r\,g\,(1 - \cos\theta). \tag{7.55}$$

Here, v is the velocity at point A ($\theta = 0^\circ$), and v' is the velocity at point B ($\theta = \theta^\circ$).

Let us now examine the radial acceleration of the object at point B. The radial forces acting on the object are the reaction R of the vertical hoop, which acts towards the centre of the hoop, and the component $m\,g\,\cos\theta$ of the object's weight, which acts away from the centre of the hoop. Since the object is executing circular motion with instantaneous tangential velocity v', it must experience an instantaneous acceleration v'^2/r towards the centre of the hoop. Hence, Newton's second law of motion yields

$$\frac{m\,v'^2}{r} = R - m\,g\,\cos\theta. \tag{7.56}$$

Note, however, that there is a constraint on the reaction R that the hoop can exert on the object. This reaction must always be *positive*. In other words, the hoop can push the object away from itself, but it can never pull it towards itself. Another way of putting this is that if the reaction ever becomes negative then the object will fly off the surface of the hoop, since it is no longer being pressed into this surface. It should be clear, by now, that the problem we are considering is exactly analogous to the earlier problem of an object attached to the end of a piece of string which is executing a vertical circle, with the reaction R of the hoop playing the role of the tension T in the string.

Let us imagine that the hoop under consideration is a "loop the loop" segment in a fairground roller-coaster. The object sliding around the inside of the loop then becomes the roller-coaster train. Suppose that the fairground operator can vary the velocity v with which the train is sent into the bottom of the loop (*i.e.*, the velocity at $\theta = 0^\circ$). What is the safe range of v? Now, if the train starts at $\theta = 0^\circ$ with velocity v then there are only *three* possible outcomes. Firstly, the train can execute a complete circuit of the loop. Secondly, the train can slide part way up the loop, come to a halt, reverse direction, and then slide back down again. Thirdly, the train can slide part way up the loop, but then fall off the loop. Obviously, it is the third possibility that the fairground operator would wish to guard against.

Using the analogy between this problem and the problem of a mass on the end of a piece of string executing a vertical circle, the condition for the roller-coaster train to execute a complete circuit is

$$v^2 > 5\,r\,g. \tag{7.57}$$

Note, interestingly enough, that this condition is independent of the mass of the train.

Equation (7.56) yields

$$v'^2 = \frac{r\,R}{m} - r\,g\,\cos\theta. \tag{7.58}$$

Now, the condition for the train to reverse direction without falling off the loop is $v'^2 = 0$ with $R > 0$. Thus, the train reverses direction when

$$R = m\,g\,\cos\theta. \tag{7.59}$$

Note that this equation can only be satisfied for positive R when $\cos\theta > 0$. In other words, the train can only turn around without falling off the loop if the

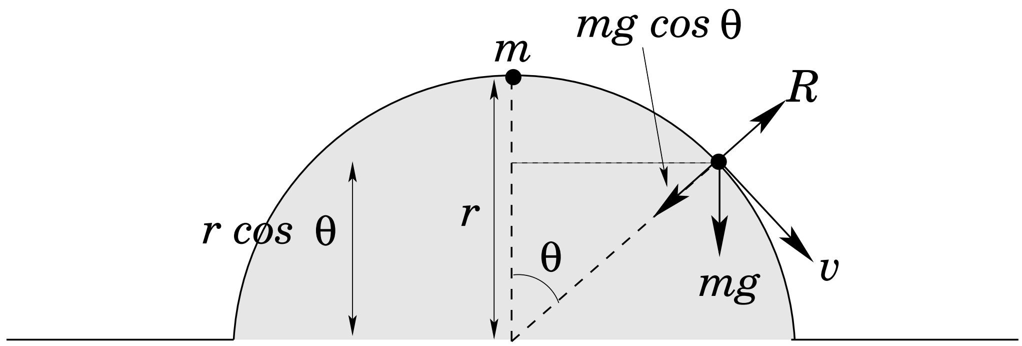


Figure 66: *A skier on a hemispherical mountain.*

turning point lies in the *lower half* of the loop (*i.e.*, $-90^\circ < \theta < 90^\circ$). The condition for the train to fall off the loop is

$$v'^2 = -r\,g\,\cos\theta. \tag{7.60}$$

Note that this equation can only be satisfied for positive v'^2 when $\cos\theta < 0$. In other words, the train can only fall off the loop when it is situated in the *upper half* of the loop. It is fairly clear that if the train's initial velocity is not sufficiently large for it to execute a complete circuit of the loop, and not sufficiently small for it to turn around before entering the upper half of the loop, then it must inevitably fall off the loop somewhere in the loop's upper half. The critical value of v^2 above which the train executes a complete circuit is $5\,r\,g$ [see Eq. (7.57)]. The critical value of v^2 at which the train just turns around before entering the upper half of the loop is $2\,r\,g$ [this is obtained from Eq. (7.55) by setting $v' = 0$ and $\theta = 90^\circ$]. Hence, the dangerous range of v^2 is

$$2\,r\,g < v^2 < 5\,r\,g. \tag{7.61}$$

For $v^2 < 2\,r\,g$, the train turns around in the lower half of the loop. For $v^2 > 5\,r\,g$, the train executes a complete circuit around the loop. However, for $2\,r\,g < v^2 < 5\,r\,g$, the train falls off the loop somewhere in its upper half.

Consider a skier of mass m skiing down a hemispherical mountain of radius r, as shown in Fig. 66. Let θ be the angular coordinate of the skier, measured with respect to the upward vertical. Suppose that the skier starts at rest ($v = 0$) on top of the mountain ($\theta = 0^\circ$), and slides down the mountain without friction. At what point does the skier fly off the surface of the mountain?

Suppose that the skier has reached angular coordinate θ. At this stage, the skier has fallen though a height $r\,(1-\cos\theta)$. Thus, the tangential velocity v of the skier is given by energy conservation:

$$\frac{1}{2}\,m\,v^2 = m\,g\,r\,(1-\cos\theta). \tag{7.62}$$

Let us now consider the skier's radial acceleration. The radial forces acting on the skier are the reaction R exerted by the mountain, which acts radially outwards, and the component of the skier's weight $m\,g\,\cos\theta$, which acts radially inwards. Since the skier is executing circular motion, radius r, with instantaneous tangential velocity v, he/she experiences an instantaneous inward radial acceleration v^2/r. Hence, Newton's second law of motion yields

$$m\,\frac{v^2}{r} = m\,g\,\cos\theta - R. \tag{7.63}$$

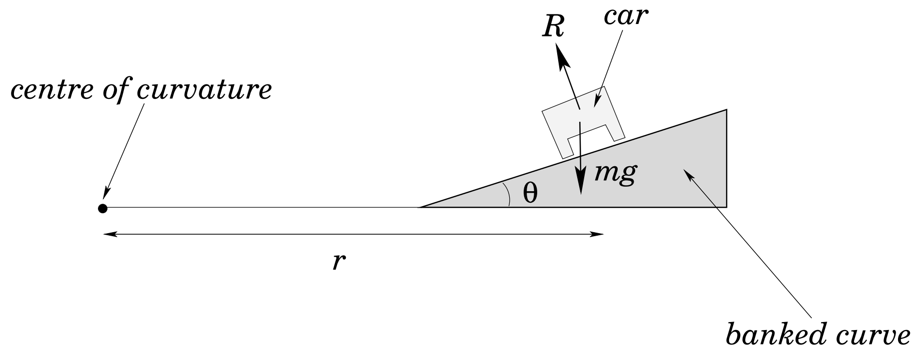
Equations (7.62) and (7.63) can be combined to give

$$R = m\,g\,(3\,\cos\theta - 2). \tag{7.64}$$

As before, the reaction R is constrained to be *positive*—the mountain can push outward on the skier, but it cannot pull the skier inward. In fact, as soon as the reaction becomes negative, the skier flies of the surface of the mountain. This occurs when $\cos\theta_0 = 2/3$, or $\theta_0 = 48.19^\circ$. The height through which the skier falls before becoming a ski-jumper is $h = r\,(1-\cos\theta_0) = a/3$.

Worked example 7.1: A banked curve

Question: Civil engineers generally *bank* curves on roads in such a manner that a car going around the curve at the recommended speed does not have to rely on friction between its tires and the road surface in order to round the curve. Suppose that the radius of curvature of a given curve is $r = 60\,\mathrm{m}$, and that the recommended speed is $v = 40\,\mathrm{km/h}$. At what angle θ should the curve be banked?



Answer: Consider a car of mass m going around the curve. The car's weight, $m\,g$, acts vertically downwards. The road surface exerts an upward normal reaction R on the car. The vertical component of the reaction must balance the downward weight of the car, so

$$R\cos\theta = m\,g.$$

The horizontal component of the reaction, $R\sin\theta$, acts towards the centre of curvature of the road. This component provides the force $m\,v^2/r$ towards the centre of the curvature which the car experiences as it rounds the curve. In other words,

$$R\sin\theta = m\,\frac{v^2}{r},$$

which yields

$$\tan\theta = \frac{v^2}{r\,g},$$

or

$$\theta = \tan^{-1}\left(\frac{v^2}{r\,g}\right).$$

Hence,

$$\theta = \tan^{-1}\left(\frac{(40\times 1000/3600)^2}{60\times 9.81}\right) = 11.8^\circ.$$

Note that if the car attempts to round the curve at the wrong speed then $m\,v^2/r \neq m\,g\tan\theta$, and the difference has to be made up by a sideways friction force exerted between the car's tires and the road surface. Unfortunately, this does not always work—especially if the road surface is wet!

Worked example 7.2: Circular race track

Question: A car of mass $m = 2000\,\mathrm{kg}$ travels around a flat circular race track of radius $r = 85\,\mathrm{m}$. The car starts at rest, and its speed increases at the constant rate $a_\theta = 0.6\,\mathrm{m/s}$. What is the speed of the car at the point when its centripetal and tangential accelerations are equal?

Answer: The tangential acceleration of the car is $a_\theta = 0.6\,\mathrm{m/s}$. When the car travels with tangential velocity v its centripetal acceleration is $a_r = v^2/r$. Hence, $a_r = a_\theta$ when

$$\frac{v^2}{r} = a_\theta,$$

or

$$v = \sqrt{r\,a_\theta} = \sqrt{85 \times 0.6} = 7.14\,\mathrm{m/s}.$$

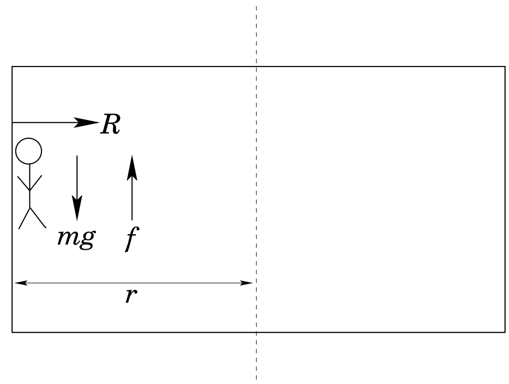
Worked example 7.3: Amusement park ride

Question: An amusement park ride consists of a vertical cylinder that spins about a vertical axis. When the cylinder spins sufficiently fast, any person inside it is held up against the wall. Suppose that the coefficient of static friction between a typical person and the wall is $\mu = 0.25$. Let the mass of an typical person be $m = 60\,\mathrm{kg}$, and let $r = 7\,\mathrm{m}$ be the radius of the cylinder. Find the critical angular velocity of the cylinder above which a typical person will not slide down the wall. How many revolutions per second is the cylinder executing at this critical velocity?

Answer: In the vertical direction, the person is subject to a downward force $m\,g$ due to gravity, and a maximum upward force $f = \mu\,R$ due to friction with the wall. Here, R is the normal reaction between the person and the wall. In order for the person not to slide down the wall, we require $f > m\,g$. Hence, the critical case corresponds to

$$f = \mu\,R = m\,g.$$

In the radial direction, the person is subject to a single force: namely, the



reaction R due to the wall, which acts radially inwards. If the cylinder (and, hence, the person) rotates with angular velocity ω, then this force must provided the acceleration $r\,\omega^2$ towards the axis of rotation. Hence,

$$R = m\,r\,\omega^2.$$

It follows that, in the critical case,

$$\omega = \sqrt{\frac{g}{\mu\,r}} = \sqrt{\frac{9.81}{0.25 \times 7}} = 2.37\,\mathrm{rad/s}.$$

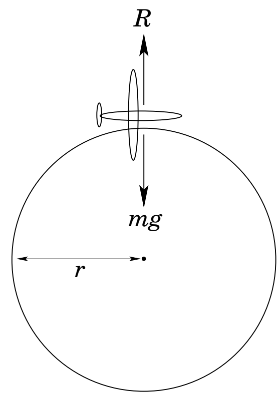
The corresponding number of revolutions per second is

$$f = \frac{\omega}{2\,\pi} = \frac{2.37}{2 \times 3.1415} = 0.38\,\mathrm{Hz}.$$

Worked example 7.4: Aerobatic maneuver

Question: A stunt pilot experiences weightlessness momentarily at the top of a "loop the loop" maneuver. Given that the speed of the stunt plane is $v = 500\,\mathrm{km/h}$, what is the radius r of the loop?

Answer: Let m be the mass of the pilot. Consider the radial acceleration of the pilot at the top of the loop. The pilot is subject to two radial forces: the gravitational force $m\,g$, which acts towards the centre of the loop, and the reaction force R, due to the plane, which acts away from the centre of the loop. Since the pilot



experiences an acceleration v^2/r towards the centre of the loop, Newton's second law of motion yields

$$m\frac{v^2}{r} = m\,g - R.$$

Now, the reaction R is equivalent to the apparent weight of the pilot. In particular, if the pilot is "weightless" then he/she exerts no force on the plane, and, therefore, the plane exerts no reaction force on the pilot. Hence, if the pilot is weightless at the top of the loop then $R = 0$, giving

$$r = \frac{v^2}{g} = \frac{(500\times 1000/3600)^2}{9.81} = 1.97\,\text{km}.$$

Worked example 7.5: Ballistic pendulum

Question: A bullet of mass $m = 10\,\text{g}$ strikes a pendulum bob of mass $M = 1.3\,\text{kg}$ horizontally with speed v, and then becomes embedded in the bob. The bob is initially at rest, and is suspended by a stiff rod of length $l = 0.6\,\text{m}$ and negligible mass. The bob is free to rotate in the vertical direction. What is the minimum value of v which causes the bob to execute a complete vertical circle? How does the answer change if the bob is suspended from a light flexible rod (of the same length), instead of a stiff rod?

Answer: When the bullet strikes the bob, and then sticks to it, the bullet and bob move off with a velocity v' which is given by momentum conservation:

$$m\,v = (M+m)\,v'.$$

Hence,

$$v' = \frac{m\,v}{M+m}.$$

Consider the case where the bob is suspended by a rigid rod. If the bob and bullet only just manage to execute a vertical loop, then their initial kinetic energy $(1/2)\,(M+m)\,v'^2$ must only just be sufficient to lift them from the bottom to the top of the loop—a distance $2\,l$. Hence, in this critical case, energy conservation yields

$$\frac{1}{2}\,(M+m)\,v'^2 = (M+m)\,2\,g\,l,$$

which implies

$$v'^2 = 4\,g\,l,$$

or

$$v = \frac{(M+m)\,\sqrt{4\,g\,l}}{m} = \frac{1.31\times\sqrt{4\times 9.81\times 0.6}}{0.01} = 635.6\,\mathrm{m/s}.$$

Consider the case where the bob is suspended by a flexible rod. The velocity v'' of the bob and bullet at the top of the loop is obtained from energy conservation:

$$\frac{1}{2}\,(M+m)\,v''^2 = \frac{1}{2}\,(M+m)\,v'^2 - (M+m)\,2\,g\,l.$$

If the bob and bullet only just manage to execute a vertical loop, then the tension in the rod is zero at the top of the loop. Hence, the acceleration due to gravity g must account exactly for the required acceleration v''^2/l towards the centre of the loop:

$$\frac{v''^2}{l} = g.$$

It follows that, in this critical case,

$$v'^2 = 5\,g\,l,$$

or

$$v = \frac{(M+m)\,\sqrt{5\,g\,l}}{m} = \frac{1.31\times\sqrt{5\times 9.81\times 0.6}}{0.01} = 710.7\,\mathrm{m/s}.$$

8 Rotational motion

8.1 Introduction

Up to now, we have only analyzed the dynamics of *point masses* (*i.e.*, objects whose spatial extent is either negligible or plays no role in their motion). Let us now broaden our approach in order to take *extended objects* into account. Now, the only type of motion which a point mass object can exhibit is *translational motion*: *i.e.*, motion by which the object moves from one point in space to another. However, an extended object can exhibit another, quite distinct, type of motion by which it remains located (more or less) at the same spatial position, but constantly changes its orientation with respect to other fixed points in space. This new type of motion is called *rotation*. Let us investigate rotational motion.

8.2 Rigid body rotation

Consider a rigid body executing pure rotational motion (*i.e.*, rotational motion which has no translational component). It is possible to define an *axis of rotation* (which, for the sake of simplicity, is assumed to pass through the body)—this axis corresponds to the straight-line which is the locus of all points inside the body which remain stationary as the body rotates. A general point located inside the body executes *circular motion* which is centred on the rotation axis, and orientated in the plane perpendicular to this axis. In the following, we tacitly assume that the axis of rotation remains fixed.

Figure 67 shows a typical rigidly rotating body. The axis of rotation is the line AB. A general point P lying within the body executes a circular orbit, centred on AB, in the plane perpendicular to AB. Let the line QP be a radius of this orbit which links the axis of rotation to the instantaneous position of P at time t. Obviously, this implies that QP is normal to AB. Suppose that at time $t + \delta t$ point P has moved to P', and the radius QP has rotated through an angle $\delta\phi$.

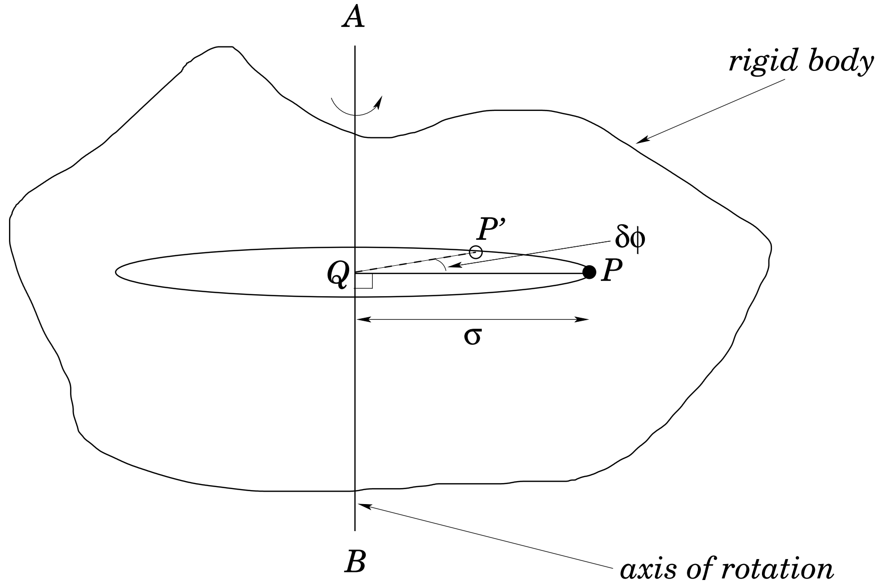


Figure 67: *Rigid body rotation.*

The instantaneous *angular velocity* of the body $\omega(t)$ is defined

$$\omega = \lim_{\delta t \to 0} \frac{\delta\phi}{\delta t} = \frac{d\phi}{dt}. \tag{8.1}$$

Note that if the body is indeed rotating rigidly, then the calculated value of ω should be the same for all possible points P lying within the body (except for those points lying exactly on the axis of rotation, for which ω is ill-defined). The rotation speed v of point P is related to the angular velocity ω of the body via

$$v = \sigma\,\omega, \tag{8.2}$$

where σ is the *perpendicular distance* from the axis of rotation to point P. Thus, in a rigidly rotating body, the rotation speed increases *linearly* with (perpendicular) distance from the axis of rotation.

It is helpful to introduce the *angular acceleration* $\alpha(t)$ of a rigidly rotating body: this quantity is defined as the time derivative of the angular velocity. Thus,

$$\alpha = \frac{d\omega}{dt} = \frac{d^2\phi}{dt^2}, \tag{8.3}$$

where ϕ is the angular coordinate of some arbitrarily chosen point reference within the body, measured with respect to the rotation axis. Note that angular velocities are conventionally measured in radians per second, whereas angular accelerations are measured in radians per second squared.

For a body rotating with constant angular velocity, ω, the angular acceleration is zero, and the rotation angle ϕ increases linearly with time:

$$\phi(t) = \phi_0 + \omega\, t, \tag{8.4}$$

where $\phi_0 = \phi(t=0)$. Likewise, for a body rotating with constant angular acceleration, α, the angular velocity increases linearly with time, so that

$$\omega(t) = \omega_0 + \alpha\, t, \tag{8.5}$$

and the rotation angle satisfies

$$\phi(t) = \phi_0 + \omega_0\, t + \frac{1}{2}\,\alpha\, t^2. \tag{8.6}$$

Here, $\omega_0 = \omega(t=0)$. Note that there is a clear analogy between the above equations, and the equations of rectilinear motion at constant acceleration introduced in Sect. 2.6—rotation angle plays the role of displacement, angular velocity plays the role of (regular) velocity, and angular acceleration plays the role of (regular) acceleration.

8.3 Is rotation a vector?

Consider a rigid body which rotates through an angle ϕ about a given axis. It is tempting to try to define a rotation "vector" $\boldsymbol{\phi}$ which describes this motion. For example, suppose that $\boldsymbol{\phi}$ is defined as the "vector" whose magnitude is the angle of rotation, ϕ, and whose direction runs parallel to the axis of rotation. Unfortunately, this definition is ambiguous, since there are two possible directions which run parallel to the rotation axis. However, we can resolve this problem by adopting the following convention—the rotation "vector" runs parallel to the axis of rotation in the sense indicated by the thumb of the right-hand, when the fingers of this hand circulate around the axis in the direction of rotation. This convention is known as the *right-hand grip* rule. The right-hand grip rule is illustrated in Fig. 68.

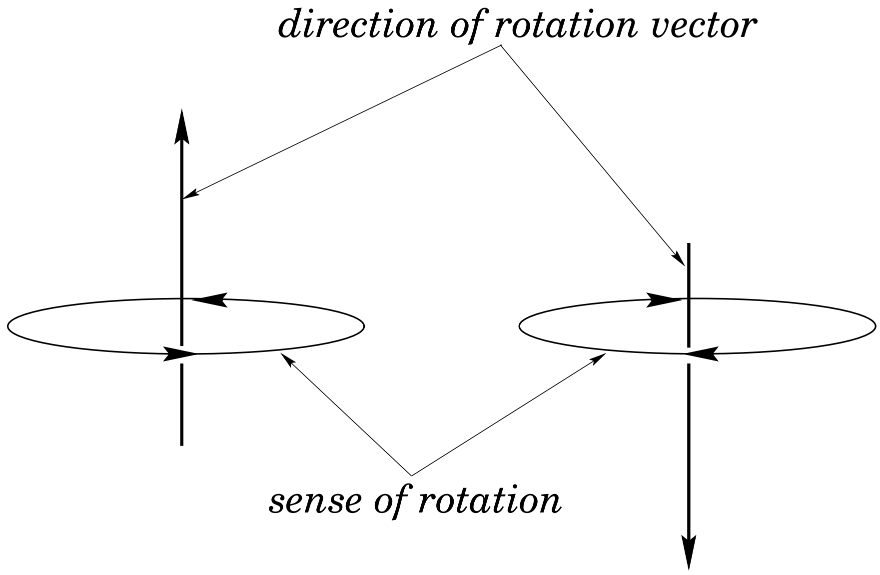


Figure 68: *The right-hand grip rule.*

The rotation "vector" $\boldsymbol{\phi}$ now has a well-defined magnitude and direction. But, is this quantity really a vector? This may seem like a strange question to ask, but it turns out that not all quantities which have well-defined magnitudes and directions are necessarily vectors. Let us review some properties of vectors. If **a** and **b** are two general vectors, then it is certainly the case that

$$\mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a}. \tag{8.7}$$

In other words, the addition of vectors is necessarily *commutative* (*i.e.*, it is independent of the order of addition). Is this true for "vector" rotations, as we have just defined them? Figure 69 shows the effect of applying two successive 90° rotations—one about the x-axis, and the other about the z-axis—to a six-sided die. In the left-hand case, the z-rotation is applied before the x-rotation, and *vice versa* in the right-hand case. It can be seen that the die ends up in two completely different states. Clearly, the z-rotation plus the x-rotation *does not* equal the x-rotation plus the z-rotation. This non-commutative algebra cannot be represented by vectors. We conclude that, although rotations have well-defined magnitudes and directions, they are *not*, in general, vector quantities.

There is a direct analogy between rotation and motion over the Earth's surface. After all, the motion of a pointer along the Earth's equator from longitude 0°W to

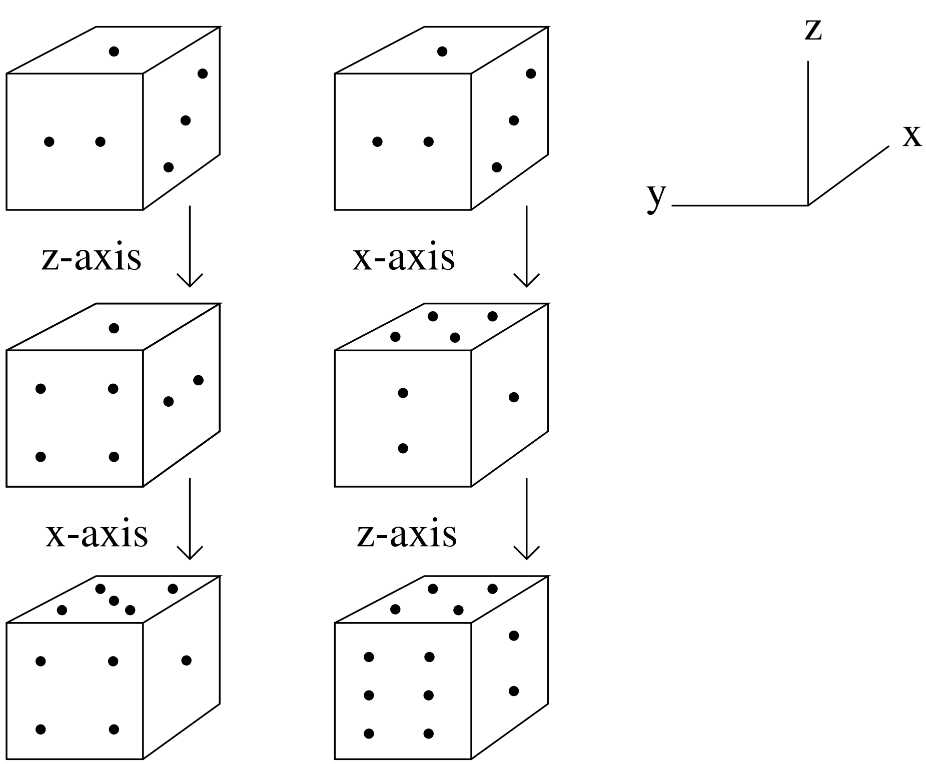


Figure 69: *The addition of rotation is non-commutative.*

longitude 90°W could just as well be achieved by keeping the pointer fixed and rotating the Earth through 90° about a North-South axis. The non-commutative nature of rotation "vectors" is a direct consequence of the non-planar (*i.e.*, curved) nature of the Earth's surface. For instance, suppose we start off at (0° N, 0° W), which is just off the Atlantic coast of equatorial Africa, and rotate 90° northwards and then 90° eastwards. We end up at (0° N, 90° E), which is in the middle of the Indian Ocean. However, if we start at the same point, and rotate 90° eastwards and then 90° northwards, we end up at the North pole. Hence, large rotations over the Earth's surface do not commute. Let us now repeat this experiment on a far smaller scale. Suppose that we walk 10 m northwards and then 10 m eastwards. Next, suppose that—starting from the same initial position—we walk 10 m eastwards and then 10 m northwards. In this case, few people would need much convincing that the two end points are essentially identical. The crucial point is that for sufficiently small displacements the Earth's surface is approximately planar, and vector displacements on a plane surface commute with one another. This observation immediately suggests that rotation "vectors" which correspond to rotations through *small angles* must also commute with one another. In other words, although the quantity $\boldsymbol{\phi}$, defined above, is not a true vector, the infinitesimal quantity $\delta\boldsymbol{\phi}$, which is defined in a similar manner but corresponds to a rotation through an infinitesimal angle $\delta\phi$, is a perfectly good vector.

We have just established that it is possible to define a true vector $\delta\boldsymbol{\phi}$ which describes a rotation through a *small* angle $\delta\phi$ about a fixed axis. But, how is this definition useful? Well, suppose that vector $\delta\boldsymbol{\phi}$ describes the small rotation that a given object executes in the infinitesimal time interval between t and $t+\delta t$. We can then define the quantity

$$\boldsymbol{\omega} = \lim_{\delta t\to 0} \frac{\delta\boldsymbol{\phi}}{\delta t} = \frac{d\boldsymbol{\phi}}{dt}. \tag{8.8}$$

This quantity is clearly a true vector, since it is simply the ratio of a true vector and a scalar. Of course, $\boldsymbol{\omega}$ represents an *angular velocity vector*. The magnitude of this vector, ω, specifies the instantaneous angular velocity of the object, whereas the direction of the vector indicates the axis of rotation. The sense of rotation is given by the right-hand grip rule: if the thumb of the right-hand points along the direction of the vector, then the fingers of the right-hand indicate the sense of rotation. We conclude that, although rotation can only be thought of as a vector quantity under certain very special circumstances, we can safely treat angular velocity as a vector quantity under all circumstances.

Suppose, for example, that a rigid body rotates at constant angular velocity $\boldsymbol{\omega}_1$. Let us now combine this motion with rotation about a *different axis* at constant angular velocity $\boldsymbol{\omega}_2$. What is the subsequent motion of the body? Since we know that angular velocity is a vector, we can be certain that the combined motion simply corresponds to rotation about a third axis at constant angular velocity

$$\boldsymbol{\omega}_3 = \boldsymbol{\omega}_1 + \boldsymbol{\omega}_2, \tag{8.9}$$

where the sum is performed according to the standard rules of vector addition. [Note, however, the following important proviso. In order for Eq. (8.9) to be valid, the rotation axes corresponding to $\boldsymbol{\omega}_1$ and $\boldsymbol{\omega}_2$ must *cross* at a certain point—the rotation axis corresponding to $\boldsymbol{\omega}_3$ then passes through this point.] Moreover, a constant angular velocity

$$\boldsymbol{\omega} = \omega_x\,\hat{\mathbf{x}} + \omega_y\,\hat{\mathbf{y}} + \omega_z\,\hat{\mathbf{z}} \tag{8.10}$$

can be thought of as representing rotation about the x-axis at angular velocity ω_x, combined with rotation about the y-axis at angular velocity ω_y, combined with rotation about the z-axis at angular velocity ω_z. [There is, again, a proviso—namely, that the rotation axis corresponding to $\boldsymbol{\omega}$ must pass through the origin. Of course, we can always shift the origin such that this is the case.] Clearly, the knowledge that angular velocity is vector quantity can be extremely useful.

8.4 The vector product

We saw earlier, in Sect. 3.10, that it is possible to combine two vectors multiplicatively, by means of a *scalar product*, to form a scalar. Recall that the scalar product $\mathbf{a}\cdot\mathbf{b}$ of two vectors $\mathbf{a} = (a_x, a_y, a_z)$ and $\mathbf{b} = (b_x, b_y, b_z)$ is defined

$$\mathbf{a}\cdot\mathbf{b} = a_x\,b_x + a_y\,b_y + a_z\,b_z = |\mathbf{a}|\,|\mathbf{b}|\,\cos\theta, \tag{8.11}$$

where θ is the angle subtended between the directions of $\mathbf{a}$ and $\mathbf{b}$.

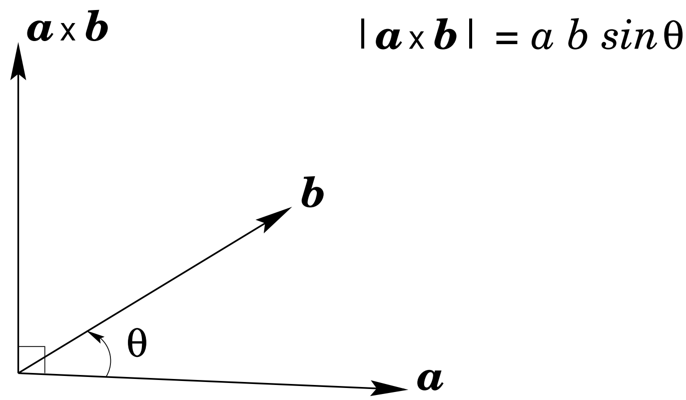


Figure 70: *The vector product.*

Is it also possible to combine two vector multiplicatively to form a third (non-coplanar) vector? It turns out that this goal can be achieved via the use of the so-called *vector product*. By definition, the vector product, $\mathbf{a} \times \mathbf{b}$, of two vectors $\mathbf{a} = (a_x, a_y, a_z)$ and $\mathbf{b} = (b_x, b_y, b_z)$ is of magnitude

$$|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}|\,|\mathbf{b}|\,\sin\theta. \tag{8.12}$$

The direction of $\mathbf{a} \times \mathbf{b}$ is *mutually perpendicular* to $\mathbf{a}$ and $\mathbf{b}$, in the sense given by the right-hand grip rule when vector $\mathbf{a}$ is rotated onto vector $\mathbf{b}$ (the direction of rotation being such that the angle of rotation is less than 180°). See Fig. 70. In coordinate form,

$$\mathbf{a} \times \mathbf{b} = (a_y\,b_z - a_z\,b_y, a_z\,b_x - a_x\,b_z, a_x\,b_y - a_y\,b_x). \tag{8.13}$$

There are a number of fairly obvious consequences of the above definition. Firstly, if vector $\mathbf{b}$ is parallel to vector $\mathbf{a}$, so that we can write $\mathbf{b} = \lambda\,\mathbf{a}$, then the vector product $\mathbf{a} \times \mathbf{b}$ has *zero magnitude*. The easiest way of seeing this is to note that if $\mathbf{a}$ and $\mathbf{b}$ are parallel then the angle θ subtended between them is zero, hence the magnitude of the vector product, $|\mathbf{a}|\,|\mathbf{b}|\,\sin\theta$, must also be zero (since $\sin 0^\circ = 0$). Secondly, the order of multiplication matters. Thus, $\mathbf{b} \times \mathbf{a}$ is *not* equivalent to $\mathbf{a} \times \mathbf{b}$. In fact, as can be seen from Eq. (8.13),

$$\mathbf{b} \times \mathbf{a} = -\mathbf{a} \times \mathbf{b}. \tag{8.14}$$

In other words, $\mathbf{b} \times \mathbf{a}$ has the same magnitude as $\mathbf{a} \times \mathbf{b}$, but points in diagrammatically the opposite direction.

Now that we have defined the vector product of two vectors, let us find a use for this concept. Figure 71 shows a rigid body rotating with angular velocity $\boldsymbol{\omega}$. For the sake of simplicity, the axis of rotation, which runs parallel to $\boldsymbol{\omega}$, is assumed to pass through the origin O of our coordinate system. Point P, whose position vector is $\mathbf{r}$, represents a general point inside the body. What is the velocity of rotation $\mathbf{v}$ at point P? Well, the magnitude of this velocity is simply

$$v = \sigma\,\omega = \omega\,r\,\sin\theta, \tag{8.15}$$

where σ is the perpendicular distance of point P from the axis of rotation, and θ is the angle subtended between the directions of $\boldsymbol{\omega}$ and $\mathbf{r}$. The direction of the velocity is *into* the page. Another way of saying this, is that the direction of the velocity is mutually perpendicular to the directions of $\boldsymbol{\omega}$ and $\mathbf{r}$, in the sense indicated by the right-hand grip rule when $\boldsymbol{\omega}$ is rotated onto $\mathbf{r}$ (through an angle less than 180°). It follows that we can write

$$\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r}. \tag{8.16}$$

Note, incidentally, that the direction of the angular velocity vector $\boldsymbol{\omega}$ indicates the orientation of the axis of rotation—however, nothing actually moves in this direction; in fact, all of the motion is *perpendicular* to the direction of $\boldsymbol{\omega}$.

8.5 Centre of mass

The *centre of mass*—or centre of gravity—of an extended object is defined in much the same manner as we earlier defined the centre of mass of a set of mutually interacting point mass objects—see Sect. 6.3. To be more exact, the coordinates of the centre of mass of an extended object are the mass weighted averages of the coordinates of the elements which make up that object. Thus, if the object has net mass M, and is composed of N elements, such that the ith element has mass m_i and position vector $\mathbf{r}_i$, then the position vector of the centre of mass is given by

$$\mathbf{r}_{cm} = \frac{1}{M}\sum_{i=1,N} m_i\,\mathbf{r}_i. \tag{8.17}$$

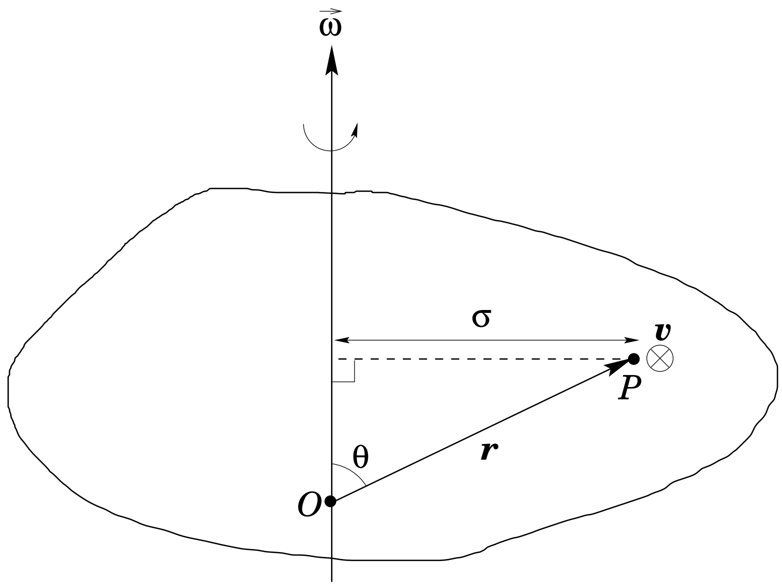


Figure 71: *Rigid rotation.*

If the object under consideration is *continuous*, then

$$m_i = \rho(\mathbf{r}_i)\,V_i, \tag{8.18}$$

where $\rho(\mathbf{r})$ is the mass density of the object, and V_i is the volume occupied by the ith element. Here, it is assumed that this volume is small compared to the total volume of the object. Taking the limit that the number of elements goes to infinity, and the volume of each element goes to zero, Eqs. (8.17) and (8.18) yield the following integral formula for the position vector of the centre of mass:

$$\mathbf{r}_{cm} = \frac{1}{M}\iiint \rho\,\mathbf{r}\,dV. \tag{8.19}$$

Here, the integral is taken over the whole volume of the object, and $dV = dx\,dy\,dz$ is an element of that volume. Incidentally, the triple integral sign indicates a volume integral: *i.e.*, a simultaneous integral over three independent Cartesian coordinates. Finally, for an object whose mass density is *constant*—which is the only type of object that we shall be considering in this course—the above expression reduces to

$$\mathbf{r}_{cm} = \frac{1}{V}\iiint \mathbf{r}\,dV, \tag{8.20}$$

where V is the volume of the object. According to Eq. (8.20), the centre of mass of a body of uniform density is located at the *geometric centre* of that body.

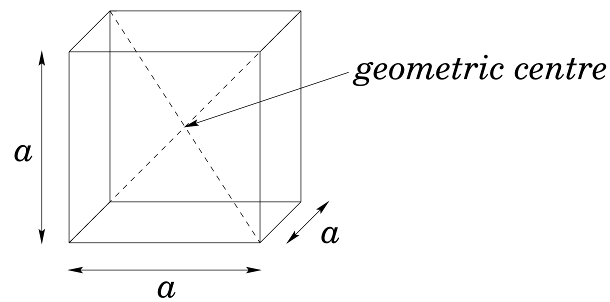


Figure 72: *Locating the geometric centre of a cube.*

For many solid objects, the location of the geometric centre follows from symmetry. For instance, the geometric centre of a cube is the point of intersection of the cube's diagonals. See Fig. 72. Likewise, the geometric centre of a right cylinder is located on the axis, half-way up the cylinder. See Fig. 73.

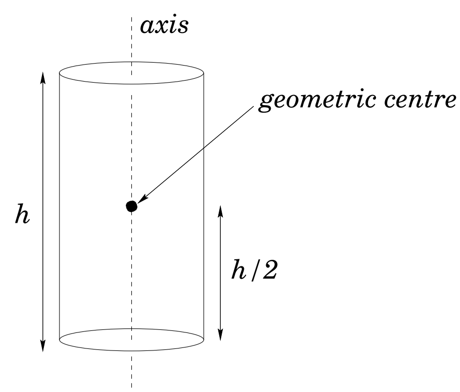


Figure 73: *Locating the geometric centre of a right cylinder.*

As an illustration of the use of formula (8.20), let us calculate the geometric centre of a regular square-sided pyramid. Figure 74 shows such a pyramid. Let a be the length of each side. It follows, from simple trigonometry, that the height of the pyramid is $h = a/\sqrt{2}$. Suppose that the base of the pyramid lies on the x-y plane, and the apex is aligned with the z-axis, as shown in the diagram. It follows, from symmetry, that the geometric centre of the pyramid lies on the z-axis. It only remains to calculate the perpendicular distance, z_{cm}, between the geometric centre and the base of the pyramid. This quantity is obtained from the

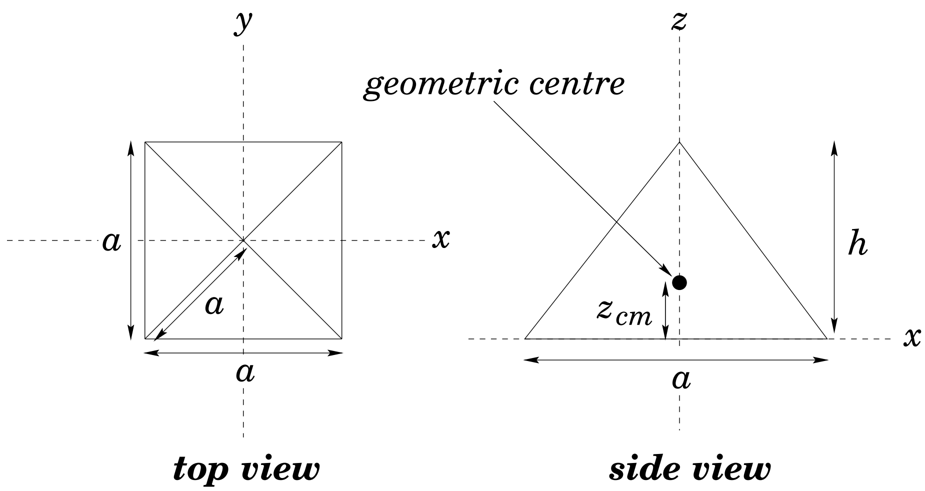


Figure 74: *Locating the geometric centre of a regular square-sided pyramid.*

z-component of Eq. (8.20):

$$z_{cm} = \frac{\int\!\int\!\int z\,dx\,dy\,dz}{\int\!\int\!\int\,dx\,dy\,dz}, \tag{8.21}$$

where the integral is taken over the volume of the pyramid.

In the above integral, the limits of integration for z are $z = 0$ to $z = h$, respectively (*i.e.*, from the base to the apex of the pyramid). The corresponding limits of integration for x and y are $x, y = -a\,(1-z/h)/2$ to $x, y = +a\,(1-z/h)/2$, respectively (*i.e.*, the limits are $x, y = \pm a/2$ at the base of the pyramid, and $x, y = \pm 0$ at the apex). Hence, Eq. (8.21) can be written more explicitly as

$$z_{cm} = \frac{\int_0^h z\,dz \int_{-a\,(1-z/h)/2}^{+a\,(1-z/h)/2} dy \int_{-a\,(1-z/h)/2}^{+a\,(1-z/h)/2} dx}{\int_0^h dz \int_{-a\,(1-z/h)/2}^{+a\,(1-z/h)/2} dy \int_{-a\,(1-z/h)/2}^{+a\,(1-z/h)/2} dx}. \tag{8.22}$$

As indicated above, it makes sense to perform the x- and y- integrals before the z-integrals, since the limits of integration for the x- and y- integrals are z-dependent. Performing the x-integrals, we obtain

$$z_{cm} = \frac{\int_0^h z\,dz \int_{-a\,(1-z/h)/2}^{+a\,(1-z/h)/2} a\,(1-z/h)\,dy}{\int_0^h dz \int_{-a\,(1-z/h)/2}^{+a\,(1-z/h)/2} a\,(1-z/h)\,dy}. \tag{8.23}$$

Performing the y-integrals, we obtain

$$z_{cm} = \frac{\int_0^h a^2\, z\,(1 - z/h)^2\,dz}{\int_0^h a^2\,(1 - z/h)^2\,dz}. \tag{8.24}$$

Finally, performing the z-integrals, we obtain

$$z_{cm} = \frac{a^2\left[z^2/2 - 2\,z^3/(3\,h) + z^4/(4\,h^2)\right]_0^h}{a^2\,[z - z^2/(h) + z^3/(3\,h)]_0^h} = \frac{a^2\,h^2/12}{a^2\,h/3} = \frac{h}{4}. \tag{8.25}$$

Thus, the geometric centre of a regular square-sided pyramid is located on the symmetry axis, one quarter of the way from the base to the apex.

8.6 Moment of inertia

Consider an extended object which is made up of N elements. Let the ith element possess mass m_i, position vector $\mathbf{r}_i$, and velocity $\mathbf{v}_i$. The total kinetic energy of the object is written

$$K = \sum_{i=1,N} \frac{1}{2}\,m_i\,v_i^2. \tag{8.26}$$

Suppose that the motion of the object consists merely of rigid rotation at angular velocity $\boldsymbol{\omega}$. It follows, from Sect. 8.4, that

$$\mathbf{v}_i = \boldsymbol{\omega} \times \mathbf{r}_i. \tag{8.27}$$

Let us write

$$\boldsymbol{\omega} = \omega\,\mathbf{k}, \tag{8.28}$$

where $\mathbf{k}$ is a unit vector aligned along the axis of rotation (which is assumed to pass through the origin of our coordinate system). It follows from the above equations that the kinetic energy of rotation of the object takes the form

$$K = \sum_{i=1,N} \frac{1}{2}\,m_i\,|\mathbf{k} \times \mathbf{r}_i|^2\,\omega^2, \tag{8.29}$$

or

$$K = \frac{1}{2}\,I\,\omega^2. \tag{8.30}$$

Here, the quantity I is termed the *moment of inertia* of the object, and is written

$$I = \sum_{i=1,N} m_i\, |\mathbf{k}\times\mathbf{r}_i|^2 = \sum_{i=1,N} m_i\, \sigma_i^2, \tag{8.31}$$

where $\sigma_i = |\mathbf{k}\times\mathbf{r}_i|$ is the perpendicular distance from the ith element to the axis of rotation. Note that for translational motion we usually write

$$K = \frac{1}{2}\, M\, v^2, \tag{8.32}$$

where M represents mass and v represents speed. A comparison of Eqs. (8.30) and (8.32) suggests that moment of inertia plays the same role in rotational motion that mass plays in translational motion.

For a continuous object, analogous arguments to those employed in Sect. 8.5 yield

$$I = \iiint \rho\, \sigma^2\, dV, \tag{8.33}$$

where $\rho(\mathbf{r})$ is the mass density of the object, $\sigma = |\mathbf{k}\times\mathbf{r}|$ is the perpendicular distance from the axis of rotation, and dV is a volume element. Finally, for an object of constant density, the above expression reduces to

$$I = M\, \frac{\iiint \sigma^2\, dV}{\iiint dV}. \tag{8.34}$$

Here, M is the total mass of the object. Note that the integrals are taken over the whole volume of the object.

The moment of inertia of a uniform object depends not only on the size and shape of that object but on the location of the axis about which the object is rotating. In particular, the same object can have different moments of inertia when rotating about different axes.

Unfortunately, the evaluation of the moment of inertia of a given body about a given axis invariably involves the performance of a nasty volume integral. In fact, there is only one trivial moment of inertia calculation—namely, the moment of inertia of a thin circular ring about a symmetric axis which runs perpendicular to the plane of the ring. See Fig. 75. Suppose that M is the mass of the ring, and

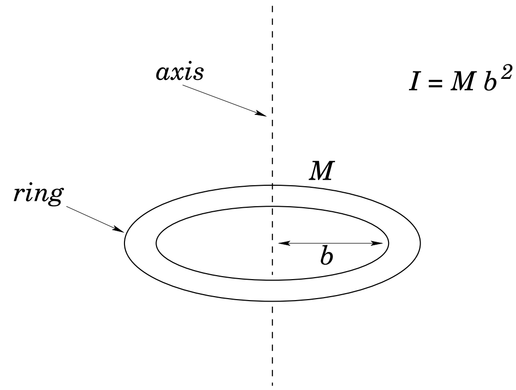


Figure 75: *The moment of inertia of a ring about a perpendicular symmetric axis.*

b is its radius. Each element of the ring shares a common perpendicular distance from the axis of rotation—*i.e.*, $\sigma = b$. Hence, Eq. (8.34) reduces to

$$I = M\,b^2. \tag{8.35}$$

In general, moments of inertia are rather tedious to calculate. Fortunately, there exist two powerful theorems which enable us to simply relate the moment of inertia of a given body about a given axis to the moment of inertia of the same body about another axis. The first of these theorems is called the *perpendicular axis theorem*, and only applies to uniform *laminar* objects. Consider a laminar object (*i.e.*, a thin, planar object) of uniform density. Suppose, for the sake of simplicity, that the object lies in the x-y plane. The moment of inertia of the object about the z-axis is given by

$$I_z = M\,\frac{\int\int (x^2+y^2)\,dx\,dy}{\int\int dx\,dy}, \tag{8.36}$$

where we have suppressed the trivial z-integration, and the integral is taken over the extent of the object in the x-y plane. Incidentally, the above expression follows from the observation that $\sigma^2 = x^2+y^2$ when the axis of rotation is coincident with the z-axis. Likewise, the moments of inertia of the object about the x- and y- axes take the form

$$I_x = M\,\frac{\int\int y^2\,dx\,dy}{\int\int dx\,dy}, \tag{8.37}$$

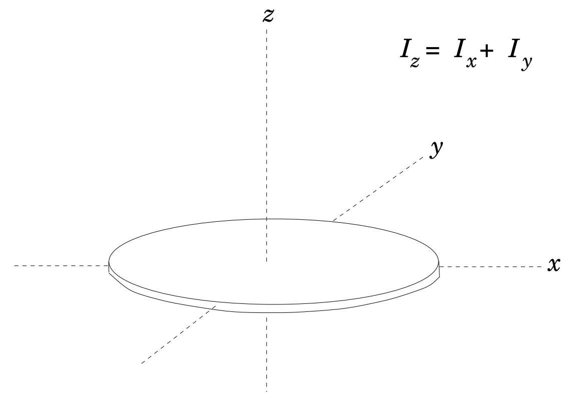


Figure 76: *The perpendicular axis theorem.*

$$I_y = M\,\frac{\int\int x^2\,dx\,dy}{\int\int dx\,dy}, \tag{8.38}$$

respectively. Here, we have made use of the fact that $z = 0$ inside the object. It follows by inspection of the previous three equations that

$$I_z = I_x + I_y. \tag{8.39}$$

See Fig. 76.

Let us use the perpendicular axis theorem to find the moment of inertia of a thin ring about a symmetric axis which lies in the plane of the ring. Adopting the coordinate system shown in Fig. 77, it is clear, from symmetry, that $I_x = I_y$. Now, we already know that $I_z = M\,b^2$, where M is the mass of the ring, and b is its radius. Hence, the perpendicular axis theorem tells us that

$$2\,I_x = I_z, \tag{8.40}$$

or

$$I_x = \frac{I_z}{2} = \frac{1}{2}\,M\,b^2. \tag{8.41}$$

Of course, $I_z > I_x$, because when the ring spins about the z-axis its elements are, on average, farther from the axis of rotation than when it spins about the x-axis.

The second useful theorem regarding moments of inertia is called the *parallel axis theorem*. The parallel axis theorem—which is quite general—states that if I is the moment of inertia of a given body about an axis passing through the centre

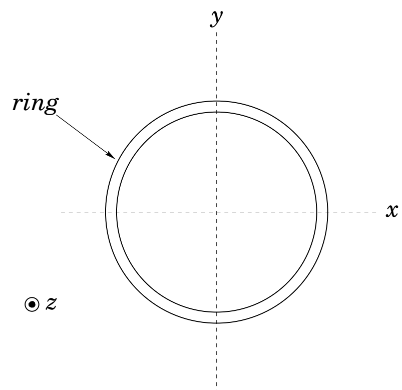


Figure 77: *The moment of inertia of a ring about a coplanar symmetric axis.*

of mass of that body, then the moment of inertia I' of the same body about a second axis which is parallel to the first is

$$I' = I + M\,d^2, \tag{8.42}$$

where M is the mass of the body, and d is the perpendicular distance between the two axes.

In order to prove the parallel axis theorem, let us choose the origin of our coordinate system to coincide with the centre of mass of the body in question. Furthermore, let us orientate the axes of our coordinate system such that the z-axis coincides with the first axis of rotation, whereas the second axis pieces the x-y plane at $x = d, y = 0$. From Eq. (8.20), the fact that the centre of mass is located at the origin implies that

$$\iiint x\,dx\,dy\,dz = \iiint y\,dx\,dy\,dz = \iiint z\,dx\,dy\,dz = 0, \tag{8.43}$$

where the integrals are taken over the volume of the body. From Eq. (8.34), the expression for the first moment of inertia is

$$I = M\,\frac{\iiint (x^2 + y^2)\;dx\,dy\,dz}{\iiint\;dx\,dy\,dz}, \tag{8.44}$$

since $x^2 + y^2$ is the perpendicular distance of a general point (x, y, z) from the z-axis. Likewise, the expression for the second moment of inertia takes the form

$$I' = M\,\frac{\iiint [(x-d)^2 + y^2]\;dx\,dy\,dz}{\iiint\;dx\,dy\,dz}. \tag{8.45}$$

The above equation can be expanded to give

$$\begin{aligned} I' &= M\,\frac{\int\!\int\!\int[(x^2+y^2)-2\,d\,x+d^2]\,dx\,dy\,dz}{\int\!\int\!\int\,dx\,dy\,dz} \\ &= M\frac{\int\!\int\!\int(x^2+y^2)\,dx\,dy\,dz}{\int\!\int\!\int\,dx\,dy\,dz}-2\,d\,M\frac{\int\!\int\!\int x\,dx\,dy\,dz}{\int\!\int\!\int\,dx\,dy\,dz} \\ &\quad +d^2\,M\frac{\int\!\int\!\int dx\,dy\,dz}{\int\!\int\!\int\,dx\,dy\,dz}. \end{aligned} \tag{8.46}$$

It follows from Eqs. (8.43) and (8.44) that

$$I' = I + M\,d^2, \tag{8.47}$$

which proves the theorem.

Let us use the parallel axis theorem to calculate the moment of inertia, I', of a thin ring about an axis which runs perpendicular to the plane of the ring, and passes through the circumference of the ring. We know that the moment of inertia of a ring of mass M and radius b about an axis which runs perpendicular to the plane of the ring, and passes through the centre of the ring—which coincides with the centre of mass of the ring—is $I = M\,b^2$. Our new axis is parallel to this original axis, but shifted sideways by the perpendicular distance b. Hence, the parallel axis theorem tells us that

$$I' = I + M\,b^2 = 2\,M\,b^2. \tag{8.48}$$

See Fig. 78.

As an illustration of the direct application of formula (8.34), let us calculate the moment of inertia of a thin circular disk, of mass M and radius b, about an axis which passes through the centre of the disk, and runs perpendicular to the plane of the disk. Let us choose our coordinate system such that the disk lies in the x-y plane with its centre at the origin. The axis of rotation is, therefore, coincident with the z-axis. Hence, formula (8.34) reduces to

$$I = M\,\frac{\int\!\int\,(x^2+y^2)\,dx\,dy}{\int\!\int\,dx\,dy}, \tag{8.49}$$

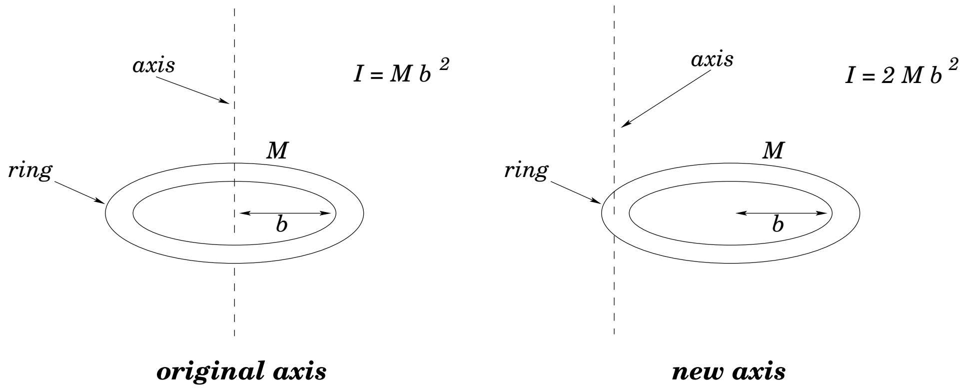


Figure 78: *An application of the parallel axis theorem.*

where the integrals are taken over the area of the disk, and the redundant z-integration has been suppressed. Let us divide the disk up into thin annuli. Consider an annulus of radius $\sigma = \sqrt{x^2 + y^2}$ and radial thickness $d\sigma$. The area of this annulus is simply $2\pi\,\sigma\,d\sigma$. Hence, we can replace $dx\,dy$ in the above integrals by $2\pi\,\sigma\,d\sigma$, so as to give

$$I = M\,\frac{\int_0^b 2\pi\,\sigma^3\,d\sigma}{\int_0^b 2\pi\,\sigma\,d\sigma}. \tag{8.50}$$

The above expression yields

$$I = M\,\frac{\left[2\,\pi\,\sigma^4/4\right]_0^b}{\left[2\,\pi\,\sigma^2/2\right]_0^b} = \frac{1}{2}\,M\,b^2. \tag{8.51}$$

Similar calculations to the above yield the following standard results:

- The moment of inertia of a thin rod of mass M and length l about an axis passing through the centre of the rod and perpendicular to its length is

$$I = \frac{1}{12}\,M\,l^2.$$

- The moment of inertia of a thin rectangular sheet of mass M and dimensions a and b about a perpendicular axis passing through the centre of the sheet is

$$I = \frac{1}{12}\,M\,(a^2 + b^2).$$

- The moment of inertia of a solid cylinder of mass M and radius b about the cylindrical axis is
$$I = \frac{1}{2}\,M\,b^2.$$
- The moment of inertia of a thin spherical shell of mass M and radius b about a diameter is
$$I = \frac{2}{3}\,M\,b^2.$$
- The moment of inertia of a solid sphere of mass M and radius b about a diameter is
$$I = \frac{2}{5}\,M\,b^2.$$

8.7 Torque

We have now identified the rotational equivalent of velocity—namely, angular velocity—and the rotational equivalent of mass—namely, moment of inertia. But, what is the rotational equivalent of force?

Consider a bicycle wheel of radius b which is free to rotate around a perpendicular axis passing through its centre. Suppose that we apply a force $\mathbf{f}$, which is coplanar with the wheel, to a point P lying on its circumference. See Fig. 79. What is the wheel's subsequent motion?

Let us choose the origin O of our coordinate system to coincide with the pivot point of the wheel—*i.e.*, the point of intersection between the wheel and the axis of rotation. Let $\mathbf{r}$ be the position vector of point P, and let θ be the angle subtended between the directions of $\mathbf{r}$ and $\mathbf{f}$. We can resolve $\mathbf{f}$ into two components—namely, a component $f\cos\theta$ which acts radially, and a component $f\sin\theta$ which acts tangentially. The radial component of $\mathbf{f}$ is canceled out by a reaction at the pivot, since the wheel is assumed to be mounted in such a manner that it can only rotate, and is prevented from displacing sideways. The tangential component of $\mathbf{f}$ causes the wheel to accelerate tangentially. Let v be the instantaneous rotation velocity of the wheel's circumference. Newton's second law of motion, applied to

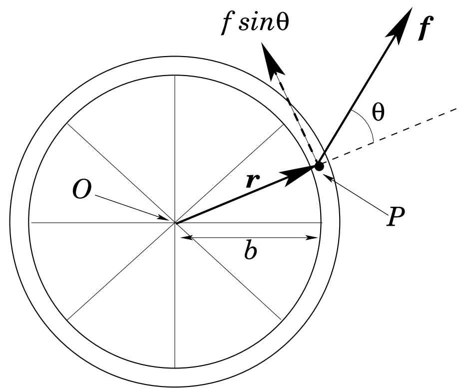


Figure 79: *A rotating bicycle wheel.*

the tangential motion of the wheel, yields

$$M\,\dot{v} = f\,\sin\theta, \tag{8.52}$$

where M is the mass of the wheel (which is assumed to be concentrated in the wheel's rim).

Let us now convert the above expression into a rotational equation of motion. If ω is the instantaneous angular velocity of the wheel, then the relation between ω and v is simply

$$v = b\,\omega. \tag{8.53}$$

Since the wheel is basically a ring of radius b, rotating about a perpendicular symmetric axis, its moment of inertia is

$$I = M\,b^2. \tag{8.54}$$

Combining the previous three equations, we obtain

$$I\,\dot{\omega} = \tau, \tag{8.55}$$

where

$$\tau = f\,b\,\sin\theta. \tag{8.56}$$

Equation (8.55) is the *angular equation of motion* of the wheel. It relates the wheel's angular velocity, ω, and moment of inertia, I, to a quantity, τ, which is known as the *torque*. Clearly, if I is analogous to mass, and ω is analogous to

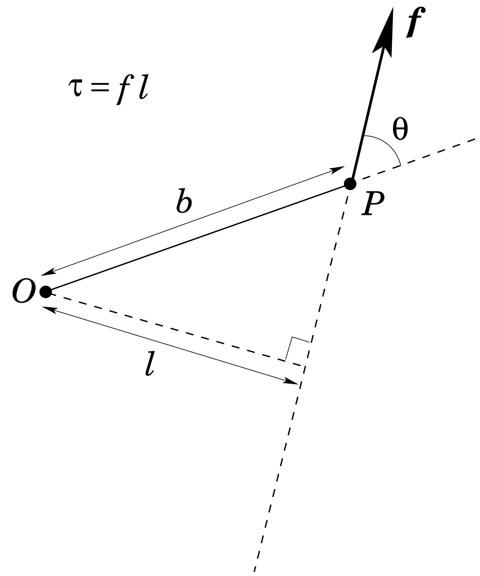


Figure 80: *Definition of the length of the level arm,* l.

velocity, then torque must be analogous to force. In other words, torque is the rotational equivalent of force.

It is clear, from Eq. (8.56), that a torque is the product of the magnitude of the applied force, f, and some distance $l = b \sin\theta$. The physical interpretation of l is illustrated in Fig. 80. If can be seen that l is the perpendicular distance of the line of action of the force from the axis of rotation. We usually refer to this distance as the length of the *lever arm*.

In summary, a torque measures the propensity of a given force to cause the object upon which it acts to twist about a certain axis. The torque, τ, is simply the product of the magnitude of the applied force, f, and the length of the lever arm, l:

$$\tau = f\,l. \tag{8.57}$$

Of course, this definition makes a lot of sense. We all know that it is far easier to turn a rusty bolt using a long, rather than a short, wrench. Assuming that we exert the same force on the end of each wrench, the torque we apply to the bolt is larger in the former case, since the perpendicular distance between the line of action of the force and the bolt (*i.e.*, the length of the wrench) is greater.

Since force is a vector quantity, it stands to reason that torque must also be a vector quantity. It follows that Eq. (8.57) defines the magnitude, τ, of some torque vector, $\boldsymbol{\tau}$. But, what is the direction of this vector? By convention, if a torque is such as to cause the object upon which it acts to twist about a certain axis, then the direction of that torque runs along the direction of the axis in the sense given by the right-hand grip rule. In other words, if the fingers of the right-hand circulate around the axis of rotation in the sense in which the torque twists the object, then the thumb of the right-hand points along the axis in the direction of the torque. It follows that we can rewrite our rotational equation of motion, Eq. (8.55), in vector form:

$$I\,\frac{d\boldsymbol{\omega}}{dt} = I\,\boldsymbol{\alpha} = \boldsymbol{\tau}, \tag{8.58}$$

where $\boldsymbol{\alpha} = d\boldsymbol{\omega}/dt$ is the vector angular acceleration. Note that the direction of $\boldsymbol{\alpha}$ indicates the direction of the rotation axis about which the object accelerates (in the sense given by the right-hand grip rule), whereas the direction of $\boldsymbol{\tau}$ indicates the direction of the rotation axis about which the torque attempts to twist the object (in the sense given by the right-hand grip rule). Of course, these two rotation axes are identical.

Although Eq. (8.58) was derived for the special case of a torque applied to a ring rotating about a perpendicular symmetric axis, it is, nevertheless, completely general.

It is important to appreciate that the directions we ascribe to angular velocities, angular accelerations, and torques are merely *conventions*. There is actually no physical motion in the direction of the angular velocity vector—in fact, all of the motion is in the plane perpendicular to this vector. Likewise, there is no physical acceleration in the direction of the angular acceleration vector—again, all of the acceleration is in the plane perpendicular to this vector. Finally, no physical forces act in the direction of the torque vector—in fact, all of the forces act in the plane perpendicular to this vector.

Consider a rigid body which is free to pivot in any direction about some fixed point O. Suppose that a force $\mathbf{f}$ is applied to the body at some point P whose position vector relative to O is $\mathbf{r}$. See Fig. 81. Let θ be the angle subtended

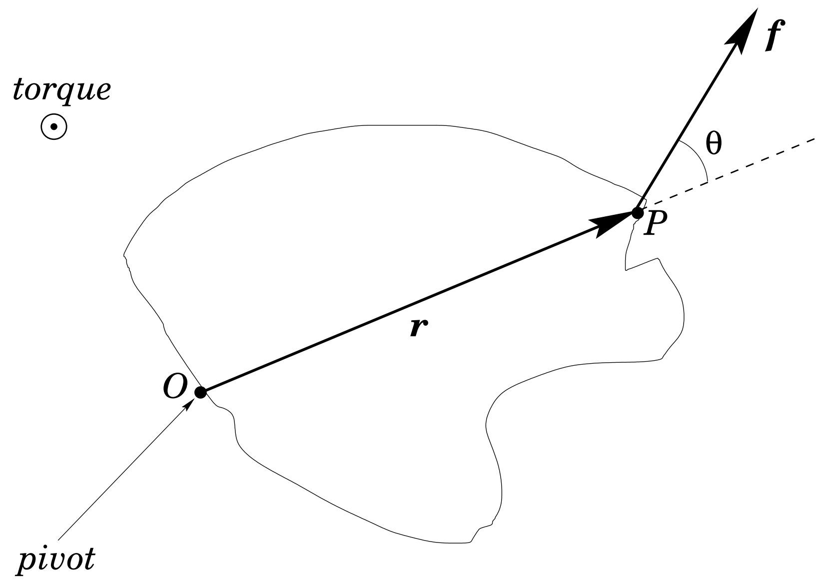


Figure 81: *Torque about a fixed point.*

between the directions of **r** and **f**. What is the vector torque $\boldsymbol{\tau}$ acting on the object about an axis passing through the pivot point? The magnitude of this torque is simply

$$\tau = r\,f\,\sin\theta. \tag{8.59}$$

In Fig. 81, the conventional direction of the torque is out of the page. Another way of saying this is that the direction of the torque is mutually perpendicular to both **r** and **f**, in the sense given by the right-hand grip rule when vector **r** is rotated onto vector **f** (through an angle less than 180° degrees). It follows that we can write

$$\boldsymbol{\tau} = \mathbf{r}\times\mathbf{f}. \tag{8.60}$$

In other words, the torque exerted by a force acting on a rigid body which pivots about some fixed point is the vector product of the displacement of the point of application of the force from the pivot point with the force itself. Equation (8.60) specifies both the magnitude of the torque, and the axis of rotation about which the torque twists the body upon which it acts. This axis runs parallel to the direction of $\boldsymbol{\tau}$, and passes through the pivot point.

8.8 Power and work

Consider a mass m attached to the end of a light rod of length l whose other end is attached to a fixed pivot. Suppose that the pivot is such that the rod is free to rotate in any direction. Suppose, further, that a force $\mathbf{f}$ is applied to the mass, whose instantaneous angular velocity about an axis of rotation passing through the pivot is $\boldsymbol{\omega}$.

Let $\mathbf{v}$ be the instantaneous velocity of the mass. We know that the rate at which the force $\mathbf{f}$ performs work on the mass—otherwise known as the power—is given by (see Sect. 5.8)

$$P = \mathbf{f}\cdot\mathbf{v}. \tag{8.61}$$

However, we also know that (see Sect. 8.4)

$$\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r}, \tag{8.62}$$

where $\mathbf{r}$ is the vector displacement of the mass from the pivot. Hence, we can write

$$P = \boldsymbol{\omega} \times \mathbf{r} \cdot \mathbf{f} \tag{8.63}$$

(note that $\mathbf{a}\cdot\mathbf{b} = \mathbf{b}\cdot\mathbf{a}$).

Now, for any three vectors, $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$, we can write

$$\mathbf{a} \times \mathbf{b} \cdot \mathbf{c} = \mathbf{a} \cdot \mathbf{b} \times \mathbf{c}. \tag{8.64}$$

This theorem is easily proved by expanding the vector and scalar products in component form using the definitions (8.11) and (8.13). It follows that Eq. (8.63) can be rewritten

$$P = \boldsymbol{\omega} \cdot \mathbf{r} \times \mathbf{f}. \tag{8.65}$$

However,

$$\boldsymbol{\tau} = \mathbf{r} \times \mathbf{f}, \tag{8.66}$$

where $\boldsymbol{\tau}$ is the torque associated with force $\mathbf{f}$ about an axis of rotation passing through the pivot. Hence, we obtain

$$P = \boldsymbol{\tau}\cdot\boldsymbol{\omega}. \tag{8.67}$$

In other words, the rate at which a torque performs work on the object upon which it acts is the scalar product of the torque and the angular velocity of the object. Note the great similarity between Eq. (8.61) and Eq. (8.67).

Now the relationship between work, W, and power, P, is simply

$$P = \frac{dW}{dt}. \tag{8.68}$$

Likewise, the relationship between angular velocity, $\boldsymbol{\omega}$, and angle of rotation, $\boldsymbol{\phi}$, is

$$\boldsymbol{\omega} = \frac{d\boldsymbol{\phi}}{dt}. \tag{8.69}$$

It follows that Eq. (8.67) can be rewritten

$$dW = \boldsymbol{\tau}\cdot d\boldsymbol{\phi}. \tag{8.70}$$

Integration yields

$$W = \int \boldsymbol{\tau}\cdot d\boldsymbol{\phi}. \tag{8.71}$$

Note that this is a good definition, since it only involves an infinitesimal rotation vector, $d\boldsymbol{\phi}$. Recall, from Sect. 8.3, that it is impossible to define a finite rotation vector. For the case of translational motion, the analogous expression to the above is

$$W = \int \mathbf{f}\cdot d\mathbf{r}. \tag{8.72}$$

Here, $\mathbf{f}$ is the force, and $d\mathbf{r}$ is an element of displacement of the body upon which the force acts.

Although Eqs. (8.67) and (8.71) were derived for the special case of the rotation of a mass attached to the end of a light rod, they are, nevertheless, completely general.

Consider, finally, the special case in which the torque is aligned with the angular velocity, and both are constant in time. In this case, the rate at which the torque performs work is simply

$$P = \tau\,\omega. \tag{8.73}$$

Translational motion		*Rotational motion*	
Displacement	$d\mathbf{r}$	Angular displacement	$d\boldsymbol{\phi}$
Velocity	$\mathbf{v} = d\mathbf{r}/dt$	Angular velocity	$\boldsymbol{\omega} = d\boldsymbol{\phi}/dt$
Acceleration	$\mathbf{a} = d\mathbf{v}/dt$	Angular acceleration	$\boldsymbol{\alpha} = d\boldsymbol{\omega}/dt$
Mass	M	Moment of inertia	$I = \int \rho \lvert\hat{\boldsymbol{\omega}}\times\mathbf{r}\rvert^2\, dV$
Force	$\mathbf{f} = M\,\mathbf{a}$	Torque	$\boldsymbol{\tau} \equiv \mathbf{r}\times\mathbf{f} = I\,\boldsymbol{\alpha}$
Work	$W = \int \mathbf{f}\cdot d\mathbf{r}$	Work	$W = \int \boldsymbol{\tau}\cdot d\boldsymbol{\phi}$
Power	$P = \mathbf{f}\cdot\mathbf{v}$	Power	$P = \boldsymbol{\tau}\cdot\boldsymbol{\omega}$
Kinetic energy	$K = M\,v^2/2$	Kinetic energy	$K = I\,\omega^2/2$

Table 3: *The analogies between translational and rotational motion.*

Likewise, the net work performed by the torque in twisting the body upon which it acts through an angle $\Delta\phi$ is just

$$W = \tau\,\Delta\phi. \tag{8.74}$$

8.9 Translational motion versus rotational motion

It should be clear, by now, that there is a strong analogy between rotational motion and standard translational motion. Indeed, each physical concept used to analyze rotational motion has its translational concomitant. Likewise, every law of physics governing rotational motion has a translational equivalent. The analogies between rotational and translational motion are summarized in Table 3.

8.10 The physics of baseball

Baseball players know from experience that there is a "sweet spot" on a baseball bat, about 17 cm from the end of the barrel, where the shock of impact with the ball, as felt by the hands, is minimized. In fact, if the ball strikes the bat exactly on the "sweet spot" then the hitter is almost unaware of the collision. Conversely, if the ball strikes the bat well away from the "sweet spot" then the impact is felt as a painful jarring of the hands.

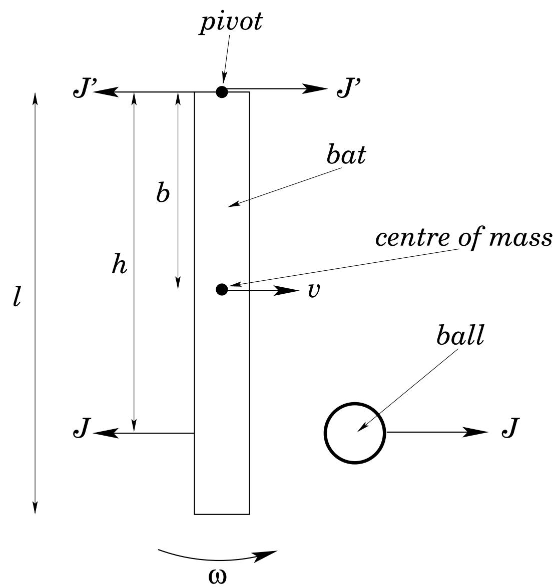


Figure 82: *A schematic baseball bat.*

The existence of a "sweet spot" on a baseball bat is just a consequence of rotational dynamics. Let us analyze this problem. Consider the schematic baseball bat shown in Fig. 82. Let M be the mass of the bat, and let l be its length. Suppose that the bat pivots about a fixed point located at one of its ends. Let the centre of mass of the bat be located a distance b from the pivot point. Finally, suppose that the ball strikes the bat a distance h from the pivot point.

The collision between the bat and the ball can be modeled as equal and opposite *impulses*, J, applied to each object at the time of the collision (see Sect. 6.5). At the same time, equal and opposite impulses J' are applied to the pivot and the bat, as shown in Fig. 82. If the pivot actually corresponds to a hitter's hands then the latter impulse gives rise to the painful jarring sensation felt when the ball is not struck properly.

We saw earlier that in a general multi-component system—which includes an extended body such as a baseball bat—the motion of the centre of mass takes a particularly simple form (see Sect. 6.3). To be more exact, the motion of the centre of mass is equivalent to that of the point particle obtained by concentrating the whole mass of the system at the centre of mass, and then allowing all of the external forces acting on the system to act upon that mass. Let us use this idea to analyze the effect of the collision with the ball on the motion of the bat's centre of mass. The centre of mass of the bat acts like a point particle of mass M which is subject to the two impulses, J and J' (which are applied simultaneously). If v is the instantaneous velocity of the centre of mass then the change in momentum of this point due to the action of the two impulses is simply

$$M\,\Delta v = -J - J'. \tag{8.75}$$

The minus signs on the right-hand side of the above equation follow from the fact that the impulses are oppositely directed to v in Fig. 82.

Note that in order to specify the instantaneous state of an extended body we must do more than just specify the location of the body's centre of mass. Indeed, since the body can rotate about its centre of mass, we must also specify its orientation in space. Thus, in order to follow the motion of an extended body, we must not only follow the translational motion of its centre of mass, but also the body's rotational motion about this point (or any other convenient reference point located within the body).

Consider the rotational motion of the bat shown in Fig. 82 about a perpendicular (to the bat) axis passing through the pivot point. This motion satisfies

$$I\,\frac{d\omega}{dt} = \tau, \tag{8.76}$$

where I is the moment of inertia of the bat, ω is its instantaneous angular velocity, and τ is the applied torque. The bat is actually subject to an impulsive torque (*i.e.*, a torque which only lasts for a short period in time) at the time of the collision with the ball. Defining the angular impulse K associated with an impulsive torque τ in much the same manner as we earlier defined the impulse associated with an impulsive force (see Sect. 6.5), we obtain

$$K = \int^{t} \tau\,dt. \tag{8.77}$$

It follows that we can integrate Eq. (8.76) over the time of the collision to find

$$I\,\Delta\omega = K, \tag{8.78}$$

where $\Delta\omega$ is the change in angular velocity of the bat due to the collision with the ball.

Now, the torque associated with a given force is equal to the magnitude of the force times the length of the lever arm. Thus, it stands to reason that the angular impulse, K, associated with an impulse, J, is simply

$$K = J\,x, \tag{8.79}$$

where x is the perpendicular distance from the line of action of the impulse to the axis of rotation. Hence, the angular impulses associated with the two impulses, J and J', to which the bat is subject when it collides with the ball, are $J\,h$ and 0, respectively. The latter angular impulse is zero since the point of application of the associated impulse coincides with the pivot point, and so the length of the lever arm is zero. It follows that Eq. (8.78) can be written

$$I\,\Delta\omega = -J\,h. \tag{8.80}$$

The minus sign comes from the fact that the impulse J is oppositely directed to the angular velocity in Fig. 82.

Now, the relationship between the instantaneous velocity of the bat's centre of mass and the bat's instantaneous angular velocity is simply

$$v = b\,\omega. \tag{8.81}$$

Hence, Eq. (8.75) can be rewritten

$$M\,b\,\Delta\omega = -J - J'. \tag{8.82}$$

Equations (8.80) and (8.82) can be combined to yield

$$J' = -\left(1 - \frac{M\,b\,h}{I}\right) J. \tag{8.83}$$

' The above expression specifies the magnitude of the impulse J' applied to the hitter's hands terms of the magnitude of the impulse J applied to the ball.

Let us crudely model the bat as a uniform rod of length l and mass M. It follows, by symmetry, that the centre of mass of the bat lies at its half-way point:

i.e.,

$$b = \frac{l}{2}. \tag{8.84}$$

Moreover, the moment of inertia of the bat about a perpendicular axis passing through one of its ends is

$$I = \frac{1}{3}\, M\, l^2 \tag{8.85}$$

(this is a standard result). Combining the previous three equations, we obtain

$$J' = -\left(1 - \frac{3\,h}{2\,l}\right) J = -\left(1 - \frac{h}{h_0}\right) J, \tag{8.86}$$

where

$$h_0 = \frac{2}{3}\, l. \tag{8.87}$$

Clearly, if $h = h_0$ then no matter how hard the ball is hit (*i.e.*, no matter how large we make J) *zero* impulse is applied to the hitter's hands. We conclude that the "sweet spot"—or, in scientific terms, the *centre of percussion*—of a uniform baseball bat lies two-thirds of the way down the bat from the hitter's end. If we adopt a more realistic model of a baseball bat, in which the bat is tapered such that the majority of its weight is located at its hitting end, we can easily demonstrate that the centre of percussion is shifted further away from the hitter (*i.e.*, it is more that two-thirds of the way along the bat).

8.11 Combined translational and rotational motion

In Sect. 4.7, we analyzed the motion of a block sliding down a frictionless incline. We found that the block accelerates down the slope with uniform acceleration $g\,\sin\theta$, where θ is the angle subtended by the incline with the horizontal. In this case, all of the potential energy lost by the block, as it slides down the slope, is converted into translational kinetic energy (see Sect. 5). In particular, no energy is dissipated.

There is, of course, no way in which a block can slide over a *frictional* surface without dissipating energy. However, we know from experience that a round object can *roll* over such a surface with hardly any dissipation. For instance, it is far easier to drag a heavy suitcase across the concourse of an airport if the suitcase has wheels on the bottom. Let us investigate the physics of round objects rolling over rough surfaces, and, in particular, rolling down rough inclines.

Consider a uniform cylinder of radius b rolling over a horizontal, frictional surface. See Fig. 83. Let v be the translational velocity of the cylinder's centre of mass, and let ω be the angular velocity of the cylinder about an axis running along its length, and passing through its centre of mass. Consider the point of contact between the cylinder and the surface. The velocity v' of this point is made up of two components: the translational velocity v, which is common to all elements of the cylinder, and the tangential velocity $v_t = -b\,\omega$, due to the cylinder's rotational motion. Thus,

$$v' = v - v_t = v - b\,\omega. \tag{8.88}$$

Suppose that the cylinder rolls *without slipping*. In other words, suppose that there is no frictional energy dissipation as the cylinder moves over the surface. This is only possible if there is zero net motion between the surface and the bottom of the cylinder, which implies $v' = 0$, or

$$v = b\,\omega. \tag{8.89}$$

It follows that when a cylinder, or any other round object, rolls across a rough surface without slipping—*i.e.*, without dissipating energy—then the cylinder's translational and rotational velocities are not independent, but satisfy a particular relationship (see the above equation). Of course, if the cylinder slips as it rolls across the surface then this relationship no longer holds.

Consider, now, what happens when the cylinder shown in Fig. 83 rolls, without slipping, down a rough slope whose angle of inclination, with respect to the horizontal, is θ. If the cylinder starts from rest, and rolls down the slope a vertical distance h, then its gravitational potential energy decreases by $-\Delta P = M\,g\,h$, where M is the mass of the cylinder. This decrease in potential energy must be offset by a corresponding increase in kinetic energy. (Recall that when a cylinder rolls without slipping there is no frictional energy loss.) However, a rolling cylinder can possesses two different types of kinetic energy. Firstly, *translational*

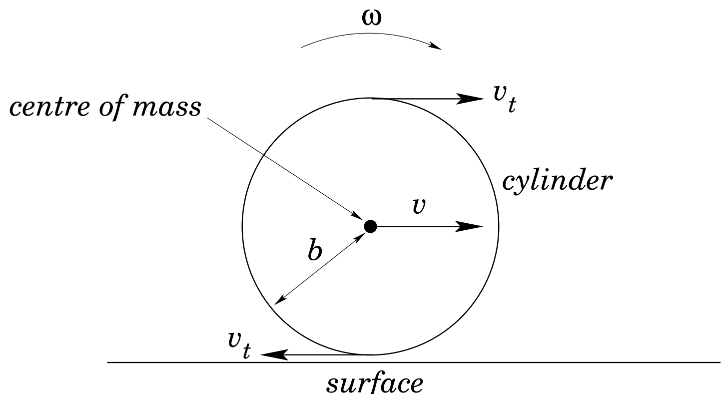


Figure 83: *A cylinder rolling over a rough surface.*

kinetic energy: $K_t = (1/2)\,M\,v^2$, where v is the cylinder's translational velocity; and, secondly, *rotational* kinetic energy: $K_r = (1/2)\,I\,\omega^2$, where ω is the cylinder's angular velocity, and I is its moment of inertia. Hence, energy conservation yields

$$M\,g\,h = \frac{1}{2}\,M\,v^2 + \frac{1}{2}\,I\,\omega^2. \tag{8.90}$$

Now, when the cylinder rolls without slipping, its translational and rotational velocities are related via Eq. (8.89). It follows from Eq. (8.90) that

$$v^2 = \frac{2\,g\,h}{1 + I/M\,b^2}. \tag{8.91}$$

Making use of the fact that the moment of inertia of a uniform cylinder about its axis of symmetry is $I = (1/2)\,M\,b^2$, we can write the above equation more explicitly as

$$v^2 = \frac{4}{3}\,g\,h. \tag{8.92}$$

Now, if the same cylinder were to slide down a *frictionless* slope, such that it fell from rest through a vertical distance h, then its final translational velocity would satisfy

$$v^2 = 2\,g\,h. \tag{8.93}$$

A comparison of Eqs. (8.92) and (8.93) reveals that when a uniform cylinder *rolls* down an incline without slipping, its final translational velocity is *less* than that obtained when the cylinder *slides* down the same incline without friction. The reason for this is that, in the former case, some of the potential energy released

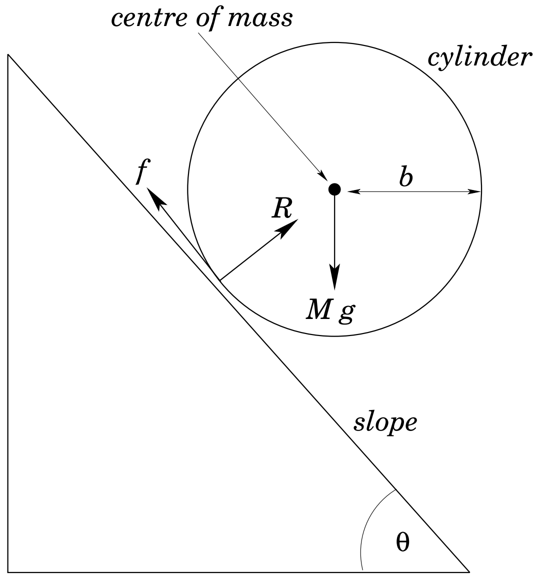


Figure 84: *A cylinder rolling down a rough incline.*

as the cylinder falls is converted into rotational kinetic energy, whereas, in the latter case, all of the released potential energy is converted into translational kinetic energy. Note that, in both cases, the cylinder's *total* kinetic energy at the bottom of the incline is equal to the released potential energy.

Let us examine the equations of motion of a cylinder, of mass M and radius b, rolling down a rough slope without slipping. As shown in Fig. 84, there are three forces acting on the cylinder. Firstly, we have the cylinder's weight, $M\,g$, which acts vertically downwards. Secondly, we have the reaction, R, of the slope, which acts normally outwards from the surface of the slope. Finally, we have the frictional force, f, which acts up the slope, parallel to its surface.

As we have already discussed, we can most easily describe the translational motion of an extended body by following the motion of its centre of mass. This motion is equivalent to that of a point particle, whose mass equals that of the body, which is subject to the same external forces as those that act on the body. Thus, applying the three forces, $M\,g$, R, and f, to the cylinder's centre of mass, and resolving in the direction normal to the surface of the slope, we obtain

$$R = M\,g\,\cos\theta. \tag{8.94}$$

Furthermore, Newton's second law, applied to the motion of the centre of mass parallel to the slope, yields

$$M\,\dot{v} = M\,g\,\sin\theta - f, \tag{8.95}$$

where $\dot{v}$ is the cylinder's translational acceleration down the slope.

Let us, now, examine the cylinder's rotational equation of motion. First, we must evaluate the torques associated with the three forces acting on the cylinder. Recall, that the torque associated with a given force is the product of the magnitude of that force and the length of the level arm—*i.e.*, the perpendicular distance between the line of action of the force and the axis of rotation. Now, by definition, the weight of an extended object acts at its centre of mass. However, in this case, the axis of rotation passes through the centre of mass. Hence, the length of the lever arm associated with the weight $M\,g$ is zero. It follows that the associated torque is also zero. It is clear, from Fig. 84, that the line of action of the reaction force, R, passes through the centre of mass of the cylinder, which coincides with the axis of rotation. Thus, the length of the lever arm associated with R is zero, and so is the associated torque. Finally, according to Fig. 84, the perpendicular distance between the line of action of the friction force, f, and the axis of rotation is just the radius of the cylinder, b—so the associated torque is $f\,b$. We conclude that the net torque acting on the cylinder is simply

$$\tau = f\,b. \tag{8.96}$$

It follows that the rotational equation of motion of the cylinder takes the form,

$$I\,\dot{\omega} = \tau = f\,b, \tag{8.97}$$

where I is its moment of inertia, and $\dot{\omega}$ is its rotational acceleration.

Now, if the cylinder rolls, without slipping, such that the constraint (8.89) is satisfied at all times, then the time derivative of this constraint implies the following relationship between the cylinder's translational and rotational accelerations:

$$\dot{v} = b\,\dot{\omega}. \tag{8.98}$$

It follows from Eqs. (8.95) and (8.97) that

$$\dot{v} = \frac{g\,\sin\theta}{1 + I/M\,b^2}, \tag{8.99}$$

$$f = \frac{M\,g\,\sin\theta}{1 + M\,b^2/I}. \tag{8.100}$$

Since the moment of inertia of the cylinder is actually $I = (1/2)\,M\,b^2$, the above expressions simplify to give

$$\dot{v} = \frac{2}{3}\,g\,\sin\theta, \tag{8.101}$$

and

$$f = \frac{1}{3}\,M\,g\,\sin\theta. \tag{8.102}$$

Note that the acceleration of a uniform cylinder as it rolls down a slope, without slipping, is only *two-thirds* of the value obtained when the cylinder slides down the same slope without friction. It is clear from Eq. (8.95) that, in the former case, the acceleration of the cylinder down the slope is retarded by friction. Note, however, that the frictional force merely acts to convert translational kinetic energy into rotational kinetic energy, and does not dissipate energy.

Now, in order for the slope to exert the frictional force specified in Eq. (8.102), without any slippage between the slope and cylinder, this force must be less than the maximum allowable static frictional force, $\mu\,R (= \mu\,M\,g\,\cos\theta)$, where μ is the coefficient of static friction. In other words, the condition for the cylinder to roll down the slope without slipping is $f < \mu\,R$, or

$$\tan\theta < 3\,\mu. \tag{8.103}$$

This condition is easily satisfied for gentle slopes, but may well be violated for extremely steep slopes (depending on the size of μ). Of course, the above condition is always violated for frictionless slopes, for which $\mu = 0$.

Suppose, finally, that we place two cylinders, side by side and at rest, at the top of a frictional slope of inclination θ. Let the two cylinders possess the same mass, M, and the same radius, b. However, suppose that the first cylinder is uniform, whereas the second is a hollow shell. Which cylinder reaches the bottom of the slope first, assuming that they are both released simultaneously, and both roll without slipping? The acceleration of each cylinder down the slope is given by Eq. (8.99). For the case of the solid cylinder, the moment of inertia is $I =$

$(1/2)\,M\,b^2$, and so

$$\dot{v}_{\rm solid} = \frac{2}{3}\,g\,\sin\theta. \tag{8.104}$$

For the case of the hollow cylinder, the moment of inertia is $I = M\,b^2$ (*i.e.*, the same as that of a ring with a similar mass, radius, and axis of rotation), and so

$$\dot{v}_{\rm hollow} = \frac{1}{2}\,g\,\sin\theta. \tag{8.105}$$

It is clear that the solid cylinder reaches the bottom of the slope *before* the hollow one (since it possesses the greater acceleration). Note that the accelerations of the two cylinders are independent of their sizes or masses. This suggests that a solid cylinder will always roll down a frictional incline faster than a hollow one, irrespective of their relative dimensions (assuming that they both roll without slipping). In fact, Eq. (8.99) suggests that whenever two different objects roll (without slipping) down the same slope, then the *most compact* object—*i.e.*, the object with the smallest $I/M\,b^2$ ratio—always wins the race.

Worked example 8.1: Balancing tires

Question: A tire placed on a balancing machine in a service station starts from rest and turns through 5.3 revolutions in 2.3 s before reaching its final angular speed. What is the angular acceleration of the tire (assuming that this quantity remains constant)? What is the final angular speed of the tire?

Answer: The tire turns through $\phi = 5.3 \times 2\,\pi = 33.30$ rad. in $t = 2.3$ s. The relationship between ϕ and t for the case of rotational motion, starting from rest, with uniform angular acceleration α is

$$\phi = \frac{1}{2}\,\alpha\,t^2.$$

Hence,

$$\alpha = \frac{2\,\phi}{t^2} = \frac{2 \times 33.30}{2.3^2} = 12.59\,{\rm rad./s^2}.$$

Given that the tire starts from rest, its angular velocity after t seconds takes the form

$$\omega = \alpha\, t = 12.59 \times 2.3 = 28.96\,\text{rad./s}.$$

Worked example 8.2: Accelerating a wheel

Question: The net work done in accelerating a wheel from rest to an angular speed of 30 rev./min. is $W = 5500\,\text{J}$. What is the moment of inertia of the wheel?

Answer: The final angular speed of the wheel is

$$\omega = 30 \times 2\,\pi/60 = 3.142\,\text{rad./s}.$$

Assuming that all of the work W performed on the wheel goes to increase its rotational kinetic energy, we have

$$W = \frac{1}{2}\, I\, \omega^2,$$

where I is the wheel's moment of inertia. It follows that

$$I = \frac{2\,W}{\omega^2} = \frac{2 \times 5500}{3.142^2} = 1114.6\,\text{kg}\,\text{m}^2.$$

Worked example 8.3: Moment of inertia of a rod

Question: A rod of mass $M = 3\,\text{kg}$ and length $L = 1.2\,\text{m}$ pivots about an axis, perpendicular to its length, which passes through one of its ends. What is the moment of inertia of the rod? Given that the rod's instantaneous angular velocity is 60 deg./s, what is its rotational kinetic energy?

Answer: The moment of inertia of a rod of mass M and length L about an axis, perpendicular to its length, which passes through its midpoint is $I = (1/12)\, M\, L^2$. This is a standard result. Using the parallel axis theorem, the moment of inertia about a parallel axis passing through one of the ends of the rod is

$$I' = I + M\left(\frac{L}{2}\right)^2 = \frac{1}{3}\, M\, L^2,$$

so

$$I' = \frac{3 \times 1.2^2}{3} = 1.44\,\mathrm{kg\,m^2}.$$

The instantaneous angular velocity of the rod is

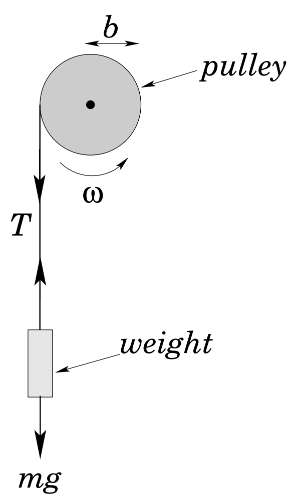
$$\omega = 60 \times \frac{\pi}{180} = 1.047\,\mathrm{rad./s}.$$

Hence, the rod's rotational kinetic energy is written

$$K = \frac{1}{2}\,I'\,\omega^2 = 0.5 \times 1.44 \times 1.047^2 = 0.789\,\mathrm{J}.$$

Worked example 8.4: Weight and pulley

Question: A weight of mass $m = 2.6\,\mathrm{kg}$ is suspended via a light inextensible cable which is wound around a pulley of mass $M = 6.4\,\mathrm{kg}$ and radius $b = 0.4\,\mathrm{m}$. Treating the pulley as a uniform disk, find the downward acceleration of the weight and the tension in the cable. Assume that the cable does not slip with respect to the pulley.



Answer: Let v be the instantaneous downward velocity of the weight, ω the instantaneous angular velocity of the pulley, and T the tension in the cable. Applying Newton's second law to the vertical motion of the weight, we obtain

$$m\,\dot{v} = m\,g - T.$$

The angular equation of motion of the pulley is written

$$I\,\dot{\omega} = \tau,$$

where I is its moment of inertia, and τ is the torque acting on the pulley. Now, the only force acting on the pulley (whose line of action does not pass through the pulley's axis of rotation) is the tension in the cable. The torque associated with this force is the product of the tension, T, and the perpendicular distance from the line of action of this force to the rotation axis, which is equal to the radius, b, of the pulley. Hence,

$$\tau = T\,b.$$

If the cable does not slip with respect to the pulley, then its downward velocity, v, must match the tangential velocity of the outer surface of the pulley, $b\,\omega$. Thus,

$$v = b\,\omega.$$

It follows that

$$\dot{v} = b\,\dot{\omega}.$$

The above equations can be combined to give

$$\dot{v} = \frac{g}{1 + I/m\,b^2},$$

$$T = \frac{m\,g}{1 + m\,b^2/I}.$$

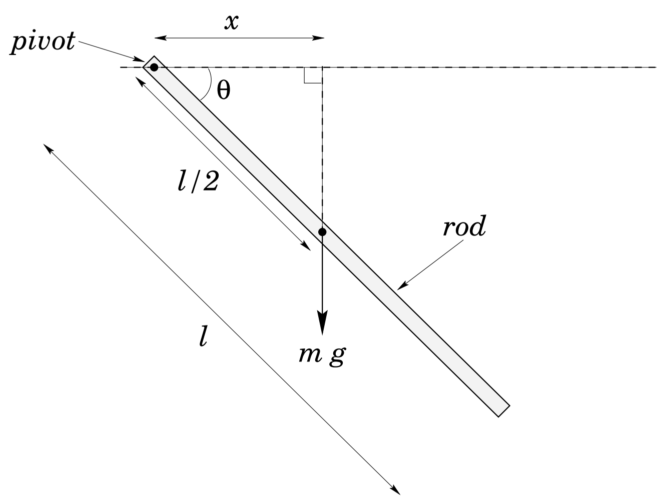
Now, the moment of inertia of the pulley is $I = (1/2)\,M\,b^2$. Hence, the above expressions reduce to

$$\dot{v} = \frac{g}{1 + M/2\,m} = \frac{9.81}{1 + 6.4/2 \times 2.6} = 4.40\,\text{m/s}^2,$$

$$T = \frac{m\,g}{1 + 2\,m/M} = \frac{2.6 \times 9.81}{1 + 2 \times 2.6/6.4} = 14.07\,\text{N}.$$

Worked example 8.5: Hinged rod

Question: A uniform rod of mass $m = 5.3\,\text{kg}$ and length $l = 1.3\,\text{m}$ rotates about a fixed frictionless pivot located at one of its ends. The rod is released from rest at an angle $\theta = 35^\circ$ beneath the horizontal. What is the angular acceleration of the rod immediately after it is released?



Answer: The moment of inertia of a rod of mass m and length l about an axis, perpendicular to its length, which passes through one of its ends is $I = (1/3)\, m\, l^2$ (see question 8.3). Hence,

$$I = \frac{5.3 \times 1.3^2}{3} = 2.986\,\text{kg m}^2.$$

The angular equation of motion of the rod is

$$I\,\alpha = \tau,$$

where α is the rod's angular acceleration, and τ is the net torque exerted on the rod. Now, the only force acting on the rod (whose line of action does not pass through the pivot) is the rod's weight, $m\,g$. This force acts at the centre of mass of the rod, which is situated at the rod's midpoint. The perpendicular distance x between the line of action of the weight and the pivot point is simply

$$x = \frac{l}{2}\cos\theta = \frac{1.3 \times \cos 35^\circ}{2} = 0.532\,\text{m}.$$

Thus, the torque acting on the rod is

$$\tau = m\,g\,x.$$

It follows that the rod's angular acceleration is written

$$\alpha = \frac{\tau}{I} = \frac{m\,g\,x}{I} = \frac{5.3 \times 9.81 \times 0.532}{2.986} = 9.26\,\text{rad./s}^2.$$

Worked example 8.6: Horsepower of engine

Question: A car engine develops a torque of $\tau = 500\,\mathrm{N\,m}$ and rotates at 3000 rev./min.. What horsepower does the engine generate? (1 hp = 746 W).

Answer: The angular speed of the engine is

$$\omega = 3000 \times 2\,\pi/60 = 314.12\,\mathrm{rad./s}.$$

Thus, the power output of the engine is

$$P = \omega\,\tau = 314.12 \times 500 = 1.57 \times 10^5\,\mathrm{W}.$$

In units of horsepower, this becomes

$$P = \frac{1.57 \times 10^5}{746} = 210.5\,\mathrm{hp}.$$

Worked example 8.7: Rotating cylinder

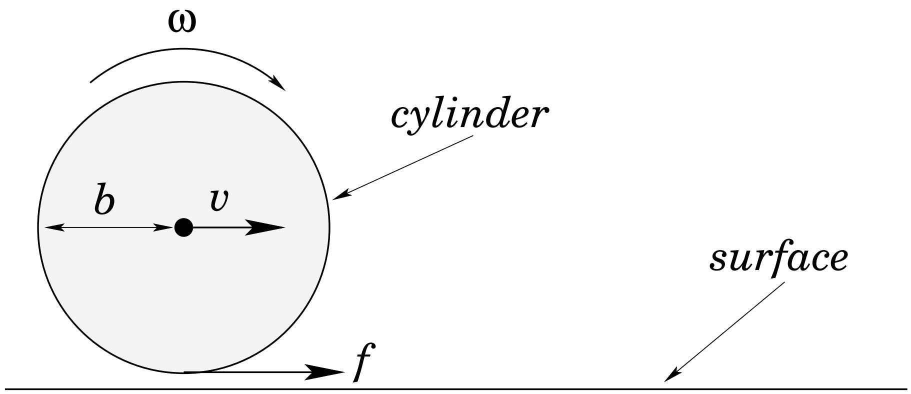
Question: A uniform cylinder of radius $b = 0.25\,\mathrm{m}$ is given an angular speed of $\omega_0 = 35\,\mathrm{rad./s}$ about an axis, parallel to its length, which passes through its centre. The cylinder is gently lowered onto a horizontal frictional surface, and released. The coefficient of friction of the surface is $\mu = 0.15$. How long does it take before the cylinder starts to roll without slipping? What distance does the cylinder travel between its release point and the point at which it commences to roll without slipping?

Answer: Let v be the velocity of the cylinder's centre of mass, ω the cylinder's angular velocity, f the frictional force exerted by the surface on the cylinder, M the cylinder's mass, and I the cylinder's moment of inertia. The cylinder's translational equation of motion is written

$$M\,\dot{v} = f.$$

Note that the friction force acts to accelerate the cylinder's translational motion. Likewise, the cylinder's rotational equation of motion takes the form

$$I\,\dot{\omega} = -f\,b,$$



since the perpendicular distance between the line of action of f and the axis of rotation is the radius, b, of the cylinder. Note that the friction force acts to decelerate the cylinder's rotational motion. If the cylinder is slipping with respect to the surface, then the friction force, f, is equal to the coefficient of friction, μ, times the normal reaction, $M\,g$, at the surface:

$$f = \mu\,M\,g.$$

Finally, the moment of inertia of the cylinder is

$$I = \frac{1}{2}\,M\,b^2.$$

The above equations can be solved to give

$$\begin{aligned}\dot{v} &= \mu\,g,\\ b\,\dot{\omega} &= -2\,\mu\,g.\end{aligned}$$

Given that $v = 0$ (*i.e.*, the cylinder is initially at rest) and $\omega = \omega_0$ at time $t = 0$, the above expressions can be integrated to give

$$\begin{aligned}v &= \mu\,g\,t,\\ b\,\omega &= b\,\omega_0 - 2\,\mu\,g\,t,\end{aligned}$$

which yields

$$v - b\,\omega = -(b\,\omega_0 - 3\,\mu\,g\,t).$$

Now, the cylinder stops slipping as soon as the "no slip" condition,

$$v = b\,\omega,$$

is satisfied. This occurs when

$$t = \frac{b\,\omega_0}{3\,\mu\,g} = \frac{0.25 \times 35}{3 \times 0.15 \times 9.81} = 1.98\,\mathrm{s}.$$

Whilst it is slipping, the cylinder travels a distance

$$x = \frac{1}{2}\,\mu\,g\,t^2 = 0.5 \times 0.15 \times 9.81 \times 1.98^2 = 2.88\,\mathrm{m}.$$

9 Angular momentum

9.1 Introduction

Two physical quantities are noticeable by their absence in Table 3. Namely, momentum, and its rotational concomitant *angular momentum*. It turns out that angular momentum is a sufficiently important concept to merit a separate discussion.

9.2 Angular momentum of a point particle

Consider a particle of mass m, position vector $\mathbf{r}$, and instantaneous velocity $\mathbf{v}$, which rotates about an axis passing through the origin of our coordinate system. We know that the particle's linear momentum is written

$$\mathbf{p} = m\,\mathbf{v}, \tag{9.1}$$

and satisfies

$$\frac{d\mathbf{p}}{dt} = \mathbf{f}, \tag{9.2}$$

where $\mathbf{f}$ is the force acting on the particle. Let us search for the rotational equivalent of $\mathbf{p}$.

Consider the quantity

$$\mathbf{l} = \mathbf{r} \times \mathbf{p}. \tag{9.3}$$

This quantity—which is known as *angular momentum*—is a vector of magnitude

$$l = r\,p\,\sin\theta, \tag{9.4}$$

where θ is the angle subtended between the directions of $\mathbf{r}$ and $\mathbf{p}$. The direction of $\mathbf{l}$ is defined to be mutually perpendicular to the directions of $\mathbf{r}$ and $\mathbf{p}$, in the sense given by the right-hand grip rule. In other words, if vector $\mathbf{r}$ rotates onto vector $\mathbf{p}$ (through an angle less than 180°), and the fingers of the right-hand are aligned with this rotation, then the thumb of the right-hand indicates the direction of $\mathbf{l}$. See Fig. 85.

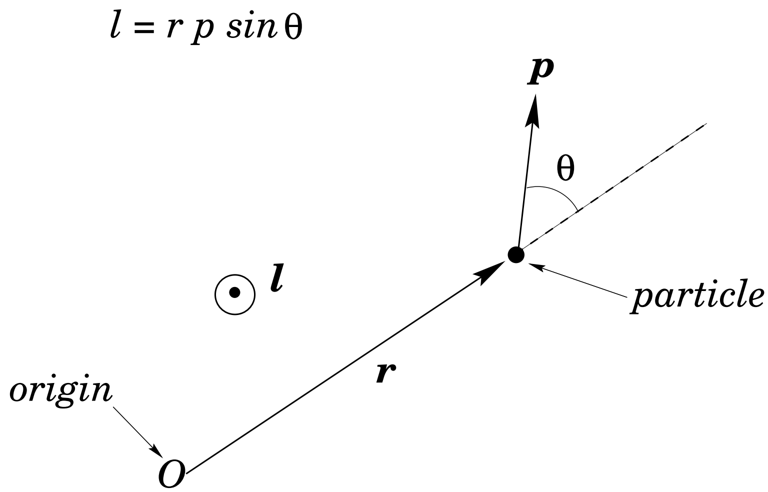


Figure 85: *Angular momentum of a point particle about the origin.*

Let us differentiate Eq. (9.3) with respect to time. We obtain

$$\frac{d\mathbf{l}}{dt} = \dot{\mathbf{r}} \times \mathbf{p} + \mathbf{r} \times \dot{\mathbf{p}}. \tag{9.5}$$

Note that the derivative of a vector product is formed in much the same manner as the derivative of an ordinary product, except that the order of the various terms is preserved. Now, we know that $\dot{\mathbf{r}} = \mathbf{v} = \mathbf{p}/m$ and $\dot{\mathbf{p}} = \mathbf{f}$. Hence, we obtain

$$\frac{d\mathbf{l}}{dt} = \frac{\mathbf{p} \times \mathbf{p}}{m} + \mathbf{r} \times \mathbf{f}. \tag{9.6}$$

However, $\mathbf{p} \times \mathbf{p} = \mathbf{0}$, since the vector product of two parallel vectors is zero. Also,

$$\mathbf{r} \times \mathbf{f} = \boldsymbol{\tau}, \tag{9.7}$$

where $\boldsymbol{\tau}$ is the torque acting on the particle about an axis passing through the origin. We conclude that

$$\frac{d\mathbf{l}}{dt} = \boldsymbol{\tau}. \tag{9.8}$$

Of course, this equation is analogous to Eq. (9.2), which suggests that angular momentum, $\mathbf{l}$, plays the role of linear momentum, $\mathbf{p}$, in rotational dynamics.

For the special case of a particle of mass m executing a *circular* orbit of radius r, with instantaneous velocity v and instantaneous angular velocity ω, the magnitude of the particle's angular momentum is simply

$$l = m\,v\,r = m\,\omega\,r^2. \tag{9.9}$$

9.3 Angular momentum of an extended object

Consider a rigid object rotating about some fixed axis with angular velocity $\boldsymbol{\omega}$. Let us model this object as a swarm of N particles. Suppose that the ith particle has mass m_i, position vector $\mathbf{r}_i$, and velocity $\mathbf{v}_i$. Incidentally, it is assumed that the object's axis of rotation passes through the origin of our coordinate system. The total angular momentum of the object, $\mathbf{L}$, is simply the vector sum of the angular momenta of the N particles from which it is made up. Hence,

$$\mathbf{L} = \sum_{i=1,N} m_i\, \mathbf{r}_i \times \mathbf{v}_i. \tag{9.10}$$

Now, for a rigidly rotating object we can write (see Sect. 8.4)

$$\mathbf{v}_i = \boldsymbol{\omega} \times \mathbf{r}_i. \tag{9.11}$$

Let

$$\boldsymbol{\omega} = \omega\, \mathbf{k}, \tag{9.12}$$

where $\mathbf{k}$ is a unit vector pointing along the object's axis of rotation (in the sense given by the right-hand grip rule). It follows that

$$\mathbf{L} = \omega \sum_{i=1,N} m_i\, \mathbf{r}_i \times (\mathbf{k} \times \mathbf{r}_i). \tag{9.13}$$

Let us calculate the component of $\mathbf{L}$ along the object's rotation axis—*i.e.*, the component along the $\mathbf{k}$ axis. We can write

$$L_k = \mathbf{L} \cdot \mathbf{k} = \omega \sum_{i=1,N} m_i\, \mathbf{k} \cdot \mathbf{r}_i \times (\mathbf{k} \times \mathbf{r}_i). \tag{9.14}$$

However, since $\mathbf{a} \cdot \mathbf{b} \times \mathbf{c} = \mathbf{a} \times \mathbf{b} \cdot \mathbf{c}$, the above expression can be rewritten

$$L_k = \omega \sum_{i=1,N} m_i\, (\mathbf{k} \times \mathbf{r}_i) \cdot (\mathbf{k} \times \mathbf{r}_i) = \omega \sum_{i=1,N} m_i\, |\mathbf{k} \times \mathbf{r}_i|^2. \tag{9.15}$$

Now,

$$\sum_{i=1,N} m_i\, |\mathbf{k} \times \mathbf{r}_i|^2 = I_k, \tag{9.16}$$

where I_k is the moment of inertia of the object about the **k** axis. (see Sect. 8.6). Hence, it follows that

$$L_k = I_k\,\omega. \tag{9.17}$$

According to the above formula, the component of a rigid body's angular momentum vector along its axis of rotation is simply the product of the body's moment of inertia about this axis and the body's angular velocity. Does this result imply that we can automatically write

$$\mathbf{L} = I\,\boldsymbol{\omega}? \tag{9.18}$$

Unfortunately, in general, the answer to the above question is *no*! This conclusion follows because the body may possess non-zero angular momentum components about axes perpendicular to its axis of rotation. Thus, in general, the angular momentum vector of a rotating body is *not* parallel to its angular velocity vector. This is a major difference from translational motion, where linear momentum is always found to be parallel to linear velocity.

For a rigid object rotating with angular velocity $\boldsymbol{\omega} = (\omega_x, \omega_y, \omega_z)$, we can write the object's angular momentum $\mathbf{L} = (L_x, L_y, L_z)$ in the form

$$L_x = I_x\,\omega_x, \tag{9.19}$$

$$L_y = I_y\,\omega_y, \tag{9.20}$$

$$L_z = I_z\,\omega_z, \tag{9.21}$$

where I_x is the moment of inertia of the object about the x-axis, *etc.* Here, it is again assumed that the origin of our coordinate system lies on the object's axis of rotation. Note that the above equations are only valid when the x-, y-, and z-axes are aligned in a certain very special manner—in fact, they must be aligned along the so-called *principal axes* of the object (these axes invariably coincide with the object's main symmetry axes). Note that it is always possible to find three, mutually perpendicular, principal axes of rotation which pass through a given point in a rigid body. Reconstructing **L** from its components, we obtain

$$\mathbf{L} = I_x\,\omega_x\,\hat{\mathbf{x}} + I_y\,\omega_y\,\hat{\mathbf{y}} + I_z\,\omega_z\,\hat{\mathbf{z}}, \tag{9.22}$$

where $\hat{\mathbf{x}}$ is a unit vector pointing along the x-axis, *etc.* It is clear, from the above equation, that the reason $\mathbf{L}$ is not generally parallel to $\boldsymbol{\omega}$ is because the moments of inertia of a rigid object about its different possible axes of rotation are *not generally the same*. In other words, if $I_x = I_y = I_z = I$ then $\mathbf{L} = I\,\boldsymbol{\omega}$, and the angular momentum and angular velocity vectors are always parallel. However, if $I_x \neq I_y \neq I_z$, which is usually the case, then $\mathbf{L}$ is not, in general, parallel to $\boldsymbol{\omega}$.

Although Eq. (9.22) suggests that the angular momentum of a rigid object is not generally parallel to its angular velocity, this equation also implies that there are, at least, *three* special axes of rotation for which this is the case. Suppose, for instance, that the object rotates about the z-axis, so that $\boldsymbol{\omega} = \omega_z\,\hat{\mathbf{z}}$. It follows from Eq. (9.22) that

$$\mathbf{L} = I_z\,\omega_z \hat{\mathbf{z}} = I_z\,\boldsymbol{\omega}. \tag{9.23}$$

Thus, in this case, the angular momentum vector is parallel to the angular velocity vector. The same can be said for rotation about the x- or y- axes. We conclude that when a rigid object rotates about one of its principal axes then its angular momentum is parallel to its angular velocity, but not, in general, otherwise.

How can we identify a principal axis of a rigid object? At the simplest level, a principal axis is one about which the object possesses *axial symmetry*. The required type of symmetry is illustrated in Fig. 86. Assuming that the object can be modeled as a swarm of particles—for every particle of mass m, located a distance r from the origin, and subtending an angle θ with the rotation axis, there must be an identical particle located on diagrammatically the opposite side of the rotation axis. As shown in the diagram, the angular momentum vectors of such a matched pair of particles can be added together to form a resultant angular momentum vector which is *parallel* to the axis of rotation. Thus, if the object is composed entirely of matched particle pairs then its angular momentum vector must be parallel to its angular velocity vector. The generalization of this argument to deal with continuous objects is fairly straightforward. For instance, symmetry implies that any axis of rotation which passes through the centre of a uniform sphere is a principal axis of that object. Likewise, a perpendicular axis which passes through the centre of a uniform disk is a principal axis. Finally, a perpendicular axis which passes through the centre of a uniform rod is a principal axis.

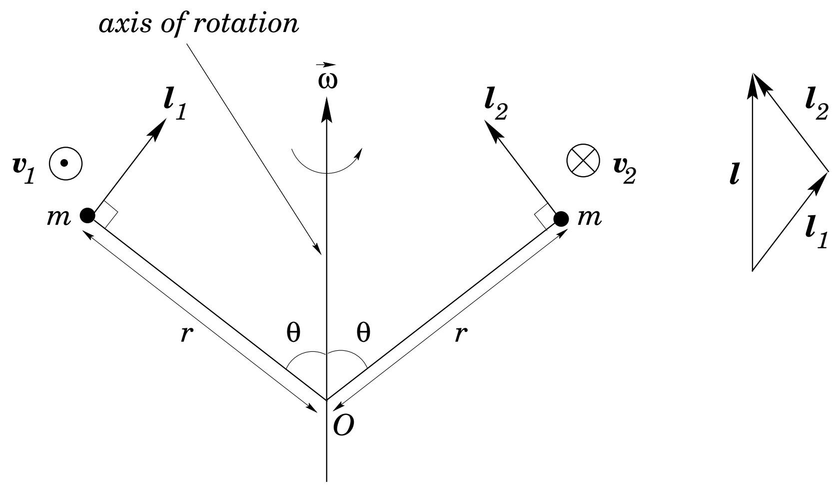


Figure 86: *A principal axis of rotation.*

9.4 Angular momentum of a multi-component system

Consider a system consisting of N mutually interacting point particles. Such a system might represent a true multi-component system, such as an asteroid cloud, or it might represent an extended body. Let the ith particle, whose mass is m_i, be located at vector displacement $\mathbf{r}_i$. Suppose that this particle exerts a force $\mathbf{f}_{ji}$ on the jth particle. By Newton's third law of motion, the force $\mathbf{f}_{ij}$ exerted by the jth particle on the ith is given by

$$\mathbf{f}_{ij} = -\mathbf{f}_{ji}. \tag{9.24}$$

Let us assume that the internal forces acting within the system are *central forces*—*i.e.*, the force $\mathbf{f}_{ij}$, acting between particles i and j, is directed along the *line of centres* of these particles. See Fig. 87. In other words,

$$\mathbf{f}_{ij} \propto (\mathbf{r}_i - \mathbf{r}_j). \tag{9.25}$$

Incidentally, this is not a particularly restrictive assumption, since most forces occurring in nature are central forces. For instance, gravity is a central force, electrostatic forces are central, and the internal stresses acting within a rigid body are approximately central. Suppose, finally, that the ith particle is subject to an external force $\mathbf{F}_i$.

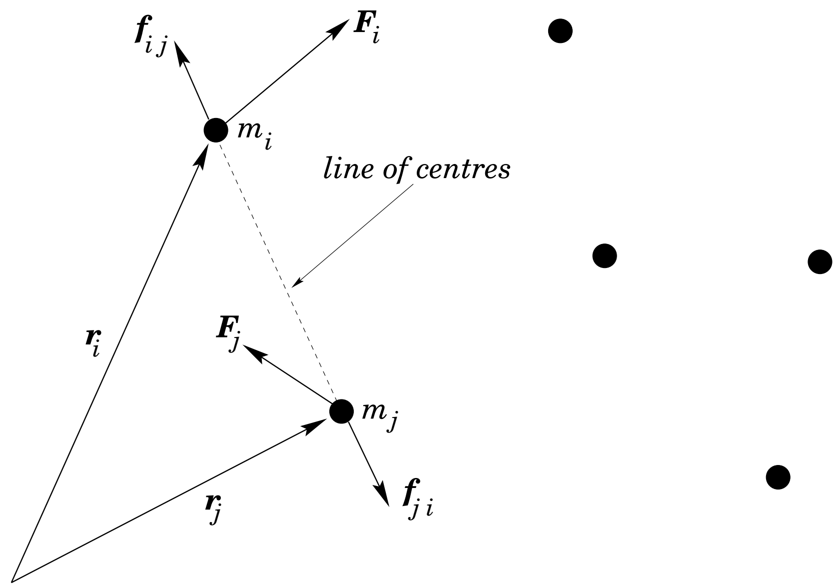


Figure 87: *A multi-component system with central internal forces.*

The equation of motion of the ith particle can be written

$$\dot{\mathbf{p}}_i = \sum_{j=1,N}^{j\neq i} \mathbf{f}_{ij} + \mathbf{F}_i. \tag{9.26}$$

Taking the vector product of this equation with the position vector $\mathbf{r}_i$, we obtain

$$\mathbf{r}_i \times \dot{\mathbf{p}}_i = \sum_{j=1,N}^{j\neq i} \mathbf{r}_i \times \mathbf{f}_{ij} + \mathbf{r}_i \times \mathbf{F}_i. \tag{9.27}$$

Now, we have already seen that

$$\mathbf{r}_i \times \dot{\mathbf{p}}_i = \frac{d(\mathbf{r}_i \times \mathbf{p}_i)}{dt}. \tag{9.28}$$

We also know that the total angular momentum, $\mathbf{L}$, of the system (about the origin) can be written in the form

$$\mathbf{L} = \sum_{i=1,N} \mathbf{r}_i \times \mathbf{p}_i. \tag{9.29}$$

Hence, summing Eq. (9.27) over all particles, we obtain

$$\frac{d\mathbf{L}}{dt} = \sum_{i,j=1,N}^{i\neq j} \mathbf{r}_i \times \mathbf{f}_{ij} + \sum_{i=1,N} \mathbf{r}_i \times \mathbf{F}_i. \tag{9.30}$$

Consider the first expression on the right-hand side of Eq. (9.30). A general term, $\mathbf{r}_i \times \mathbf{f}_{ij}$, in this sum can always be paired with a matching term, $\mathbf{r}_j \times \mathbf{f}_{ji}$, in which the indices have been swapped. Making use of Eq. (9.24), the sum of a general matched pair can be written

$$\mathbf{r}_i \times \mathbf{f}_{ij} + \mathbf{r}_j \times \mathbf{f}_{ji} = (\mathbf{r}_i - \mathbf{r}_j) \times \mathbf{f}_{ij}. \tag{9.31}$$

However, if the internal forces are central in nature then $\mathbf{f}_{ij}$ is parallel to $(\mathbf{r}_i - \mathbf{r}_j)$. Hence, the vector product of these two vectors is zero. We conclude that

$$\mathbf{r}_i \times \mathbf{f}_{ij} + \mathbf{r}_j \times \mathbf{f}_{ji} = \mathbf{0}, \tag{9.32}$$

for any values of i and j. Thus, the first expression on the right-hand side of Eq. (9.30) sums to zero. We are left with

$$\frac{d\mathbf{L}}{dt} = \boldsymbol{\tau}, \tag{9.33}$$

where

$$\boldsymbol{\tau} = \sum_{i=1,N} \mathbf{r}_i \times \mathbf{F}_i \tag{9.34}$$

is the net external torque acting on the system (about an axis passing through the origin). Of course, Eq. (9.33) is simply the rotational equation of motion for the system taken as a whole.

Suppose that the system is *isolated*, such that it is subject to *zero net external torque*. It follows from Eq. (9.33) that, in this case, *the total angular momentum of the system is a conserved quantity*. To be more exact, the components of the total angular momentum taken about any three independent axes are individually conserved quantities. Conservation of angular momentum is an extremely useful concept which greatly simplifies the analysis of a wide range of rotating systems. Let us consider some examples.

Suppose that two identical weights of mass m are attached to a light rigid rod which rotates without friction about a perpendicular axis passing through its midpoint. Imagine that the two weights are equipped with small motors which allow them to travel along the rod: the motors are synchronized in such a manner that the distance of the two weights from the axis of rotation is always the same. Let

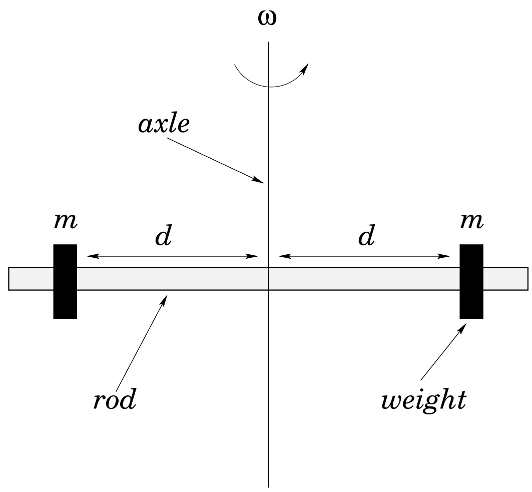


Figure 88: *Two movable weights on a rotating rod.*

us call this common distance d, and let ω be the angular velocity of the rod. See Fig. 88. How does the angular velocity ω change as the distance d is varied?

Note that there are no external torques acting on the system. It follows that the system's angular momentum must remain constant as the weights move along the rod. Neglecting the contribution of the rod, the moment of inertia of the system is written

$$I = 2\,m\,d^2. \tag{9.35}$$

Since the system is rotating about a principal axis, its angular momentum takes the form

$$L = I\,\omega = 2\,m\,d^2\,\omega. \tag{9.36}$$

If L is a constant of the motion then we obtain

$$\omega\,d^2 = \text{constant}. \tag{9.37}$$

In other words, the system spins *faster* as the weights move *inwards* towards the axis of rotation, and *vice versa*. This effect is familiar from figure skating. When a skater spins about a vertical axis, her angular momentum is approximately a conserved quantity, since the ice exerts very little torque on her. Thus, if the skater starts spinning with outstretched arms, and then draws her arms inwards, then her rate of rotation will spontaneously increase in order to conserve angular momentum. The skater can slow her rate of rotation by simply pushing her arms outwards again.

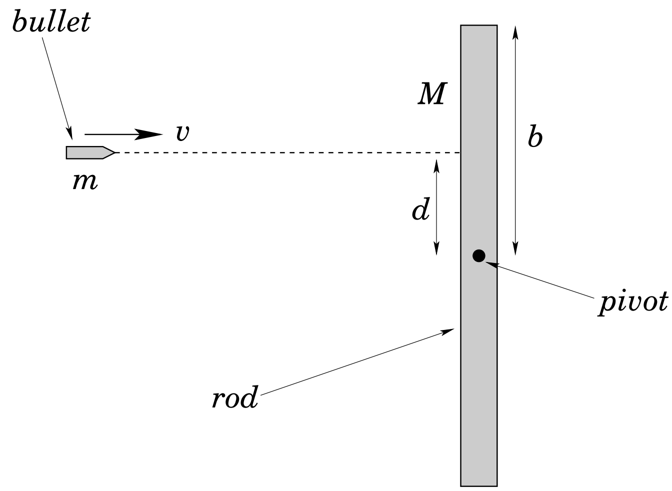


Figure 89: *A bullet strikes a pivoted rod.*

Suppose that a bullet of mass m and velocity v strikes, and becomes embedded in, a stationary rod of mass M and length $2\,b$ which pivots about a frictionless perpendicular axle passing through its mid-point. Let the bullet strike the rod normally a distance d from its axis of rotation. See Fig. 89. What is the instantaneous angular velocity ω of the rod (and bullet) immediately after the collision?

Taking the bullet and the rod as a whole, this is again a system upon which no external torque acts. Thus, we expect the system's net angular momentum to be the same before and after the collision. Before the collision, only the bullet possesses angular momentum, since the rod is at rest. As is easily demonstrated, the bullet's angular momentum about the pivot point is

$$l = m\,v\,d: \tag{9.38}$$

i.e., the product of its mass, its velocity, and its distance of closest approach to the point about which the angular momentum is measured—this is a general result (for a point particle). After the collision, the bullet lodges a distance d from the pivot, and is forced to co-rotate with the rod. Hence, the angular momentum of the bullet after the collision is given by

$$l' = m\,d^2\,\omega, \tag{9.39}$$

where ω is the angular velocity of the rod. The angular momentum of the rod after the collision is

$$L = I\,\omega, \tag{9.40}$$

where $I = (1/12)\,M\,(2\,b)^2 = (1/3)\,M\,b^2$ is the rod's moment of inertia (about a perpendicular axis passing through its mid-point). Conservation of angular momentum yields

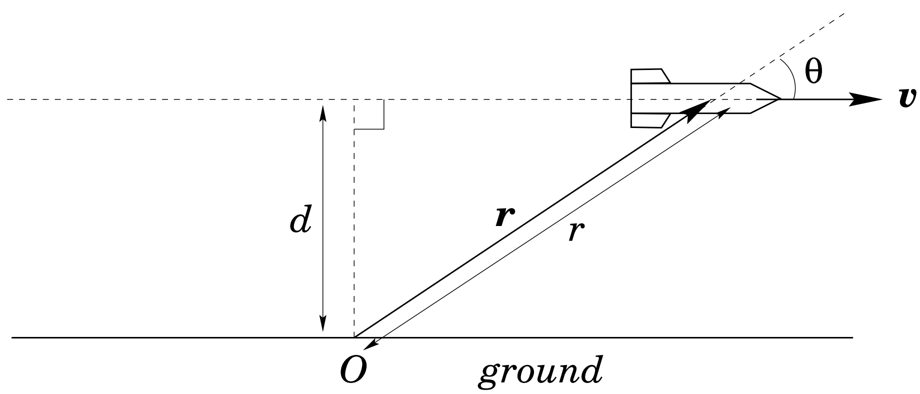
$$l = l' + L, \tag{9.41}$$

or

$$\omega = \frac{m\,v\,d}{I + m\,d^2}. \tag{9.42}$$

Worked example 9.1: Angular momentum of a missile

Question: A missile of mass $m = 2.3\times 10^4$ kg flies level to the ground at an altitude of $d = 10,000$ m with constant speed $v = 210$ m/s. What is the magnitude of the missile's angular momentum relative to a point on the ground directly below its flight path?



Answer: The missile's angular momentum about point O is

$$L = m\,v\,r\,\sin\theta,$$

where θ is the angle subtended between the missile's velocity vector and its position vector relative to O. However,

$$r\,\sin\theta = d,$$

where d is the distance of closest approach of the missile to point O. Hence,

$$L = m\,v\,d = (2.3\times 10^4)\times 210\times (1\times 10^4) = 4.83\times 10^{10}\,\text{kg}\,\text{m}^2/\text{s}.$$

Worked example 9.2: Angular momentum of a sphere

Question: A uniform sphere of mass $M = 5\,\mathrm{kg}$ and radius $a = 0.2\,\mathrm{m}$ spins about an axis passing through its centre with period $T = 0.7\,\mathrm{s}$. What is the angular momentum of the sphere?

Answer: The angular velocity of the sphere is

$$\omega = \frac{2\pi}{T} = \frac{2\pi}{0.7} = 8.98\,\mathrm{rad./s}.$$

The moment of inertia of the sphere is

$$I = \frac{2}{5}\,M\,a^2 = 0.4 \times 5 \times (0.2)^2 = 0.08\,\mathrm{kg\,m^2}.$$

Hence, the angular momentum of the sphere is

$$L = I\,\omega = 0.08 \times 8.98 = 0.718\,\mathrm{kg\,m^2/s}.$$

Worked example 9.3: Spinning skater

Question: A skater spins at an initial angular velocity of $\omega_1 = 11\,\mathrm{rad./s}$ with her arms outstretched. The skater then lowers her arms, thereby decreasing her moment of inertia by a factor 8. What is the skater's final angular velocity? Assume that any friction between the skater's skates and the ice is negligible.

Answer: Neglecting any friction between the skates and the ice, we expect the skater to spin with constant angular momentum. The skater's initial angular momentum is

$$L_1 = I_1\,\omega_1,$$

where I_1 is the skater's initial moment of inertia. The skater's final angular momentum is

$$L_2 = I_2\,\omega_2,$$

where I_2 is the skater's final moment of inertia, and ω_2 is her final angular velocity. Conservation of angular momentum yields $L_1 = L_2$, or

$$\omega_2 = \frac{I_1}{I_2}\,\omega_2.$$

Now, we are told that $I_1/I_2 = 8$. Hence,

$$\omega_2 = 8 \times 11 = 88\,\text{rad./s.}$$

10 Statics

10.1 Introduction

Probably the most useful application of the laws of mechanics is the study of situations in which nothing moves—this discipline is known as *statics*. The principles of statics are employed by engineers whenever they design stationary structures, such as buildings, bridges, and tunnels, in order to ensure that these structures do not collapse.

10.2 The principles of statics

Consider a general extended body which is subject to a number of external forces. Let us model this body as a swarm of N point particles. In the limit that $N \to \infty$, this model becomes a fully accurate representation of the body's dynamics.

In Sect. 6.3 we determined that the overall translational equation of motion of a general N-component system can be written in the form

$$\frac{d\mathbf{P}}{dt} = \mathbf{F}. \tag{10.1}$$

Here, $\mathbf{P}$ is the total linear momentum of the system, and

$$\mathbf{F} = \sum_{i=1,N} \mathbf{F}_i \tag{10.2}$$

is the resultant of all the external forces acting on the system. Note that $\mathbf{F}_i$ is the external force acting on the ith component of the system.

Equation (10.1) effectively determines the *translational motion* of the system's centre of mass. Note, however, that in order to fully determine the motion of the system we must also follow its *rotational motion* about its centre of mass (or any other convenient reference point). In Sect. 9.4 we determined that the overall rotational equation of motion of a general N-component system (with central internal forces) can be written in the form

$$\frac{d\mathbf{L}}{dt} = \boldsymbol{\tau}. \tag{10.3}$$

Here, $\mathbf{L}$ is the total angular momentum of the system (about the origin of our coordinate scheme), and

$$\boldsymbol{\tau} = \sum_{i=1,N} \mathbf{r}_i \times \mathbf{F}_i \tag{10.4}$$

is the resultant of all the external torques acting on the system (about the origin of our coordinate scheme). In the above, $\mathbf{r}_i$ is the vector displacement of the ith component of the system.

What conditions must be satisfied by the various external forces and torques acting on the system if it is to remain stationary in time? Well, if the system does not evolve in time then its net linear momentum, $\mathbf{P}$, and its net angular momentum, $\mathbf{L}$, must both remain constant. In other words, $d\mathbf{P}/dt = d\mathbf{L}/dt = \mathbf{0}$. It follows from Eqs. (10.1) and (10.3) that

$$\mathbf{F} = \mathbf{0}, \tag{10.5}$$

$$\boldsymbol{\tau} = \mathbf{0}. \tag{10.6}$$

In other words, the net external force acting on system must be zero, and the net external torque acting on the system must be zero. To be more exact:

> The components of the net external force acting along any three independent directions must all be zero;

and

> The magnitudes of the net external torques acting about any three independent axes (passing through the origin of the coordinate system) must all be zero.

In a nutshell, these are the principles of statics.

It is clear that the above principles are *necessary* conditions for a general physical system not to evolve in time. But, are they also *sufficient* conditions? In other words, is it necessarily true that a general system which satisfies these conditions does not exhibit any time variation? The answer to this question is as follows: if the system under investigation is a *rigid body*, such that the motion of any component of the body necessarily implies the motion of the whole body, then the above principles are necessary and sufficient conditions for the existence of an equilibrium state. On the other hand, if the system is not a rigid body, so that some components of the body can move independently of others, then the above conditions only guarantee that the system remains static in an average sense.

Before we attempt to apply the principles of statics, there are a couple of important points which need clarification. Firstly, does it matter about which point we calculate the net torque acting on the system? To be more exact, if we determine that the net torque acting about a given point is zero does this necessarily imply that the net torque acting about any other point is also zero? Well,

$$\boldsymbol{\tau} = \sum_{i=1,N} \mathbf{r}_i \times \mathbf{F}_i \tag{10.7}$$

is the net torque acting on the system about the origin of our coordinate scheme. The net torque about some general point $\mathbf{r}_0$ is simply

$$\boldsymbol{\tau}' = \sum_{i=1,N} (\mathbf{r}_i - \mathbf{r}_0) \times \mathbf{F}_i. \tag{10.8}$$

However, we can rewrite the above expression as

$$\boldsymbol{\tau}' = \sum_{i=1,N} \mathbf{r}_i \times \mathbf{F}_i - \mathbf{r}_0 \times \left(\sum_{i=1,N} \mathbf{F}_i \right) = \boldsymbol{\tau} + \mathbf{r}_0 \times \mathbf{F}. \tag{10.9}$$

Now, if the system is in equilibrium then $\mathbf{F} = \boldsymbol{\tau} = \mathbf{0}$. Hence, it follows from the above equation that

$$\boldsymbol{\tau}' = \mathbf{0}. \tag{10.10}$$

In other words, for a system in *equilibrium*, the determination that the net torque acting about a given point is zero necessarily implies that the net torque acting about any other point is also zero. Hence, we can choose the point about which we calculate the net torque at will—this choice is usually made so as to simplify the calculation.

Another question which needs clarification is as follows. At which point should we assume that the weight of the system acts in order to calculate the contribution of the weight to the net torque acting about a given point? Actually, in Sect. 8.11, we effectively answered this question by assuming that the weight acts at the centre of mass of the system. Let us now justify this assumption. The external force acting on the ith component of the system due to its weight is

$$\mathbf{F}_i = m_i\,\mathbf{g}, \tag{10.11}$$

where $\mathbf{g}$ is the acceleration due to gravity (which is assumed to be uniform throughout the system). Hence, the net gravitational torque acting on the system about the origin of our coordinate scheme is

$$\boldsymbol{\tau} = \sum_{i=1,N} \mathbf{r}_i \times m_i\,\mathbf{g} = \left(\sum_{i=1,N} m_i\,\mathbf{r}_i\right) \times \mathbf{g} = \mathbf{r}_{cm} \times M\,\mathbf{g}, \tag{10.12}$$

where $M = \sum_{i=1,N} m_i$ is the total mass of the system, and $\mathbf{r}_{cm} = \sum_{i=1,N} m_i\,\mathbf{r}_i/M$ is the position vector of its centre of mass. It follows, from the above equation, that the net gravitational torque acting on the system about a given point can be calculated by assuming that the total mass of the system is concentrated at its centre of mass.

10.3 Equilibrium of a laminar object in a gravitational field

Consider a general laminar object which is free to pivot about a fixed perpendicular axis. Assuming that the object is placed in a uniform gravitational field (such as that on the surface of the Earth), what is the object's equilibrium configuration in this field?

Let O represent the pivot point, and let C be the centre of mass of the object. See Fig. 90. Suppose that r represents the distance between points O and C, whereas θ is the angle subtended between the line OC and the downward

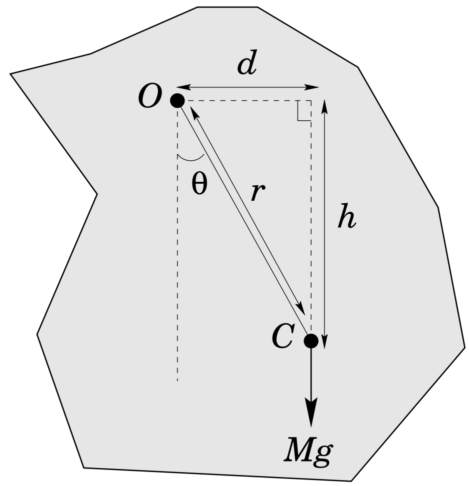


Figure 90: *A laminar object pivoting about a fixed point in a gravitational field.*

vertical. There are two external forces acting on the object. First, there is the downward force, $M\,g$, due to gravity, which acts at the centre of mass. Second, there is the reaction, R, due to the pivot, which acts at the pivot point. Here, M is the mass of the object, and g is the acceleration due to gravity.

Two conditions must be satisfied in order for a given configuration of the object shown in Fig. 90 to represent an equilibrium configuration. First, there must be zero net external force acting on the object. This implies that the reaction, R, is equal and opposite to the gravitational force, $M\,g$. In other words, the reaction is of magnitude $M\,g$ and is directed vertically upwards. The second condition is that there must be zero net torque acting about the pivot point. Now, the reaction, R, does not generate a torque, since it acts at the pivot point. Moreover, the torque associated with the gravitational force, $M\,g$, is simply the magnitude of this force times the length of the lever arm, d (see Fig. 90). Hence, the net torque acting on the system about the pivot point is

$$\tau = M\,g\,d = M\,g\,r\,\sin\theta. \tag{10.13}$$

Setting this torque to zero, we obtain $\sin\theta = 0$, which implies that $\theta = 0^\circ$. In other words, the equilibrium configuration of a general laminar object (which is free to rotate about a fixed perpendicular axis in a uniform gravitational field) is that in which the centre of mass of the object is aligned *vertically below* the pivot point.

Incidentally, we can use the above result to experimentally determine the centre of mass of a given laminar object. We would need to suspend the object from two different pivot points, successively. In each equilibrium configuration, we would mark a line running vertically downward from the pivot point, using a plumb-line. The crossing point of these two lines would indicate the position of the centre of mass.

Our discussion of the equilibrium configuration of the laminar object shown in Fig. 90 is not quite complete. We have determined that the condition which must be satisfied by an equilibrium state is $\sin\theta = 0$. However, there are, in fact, *two* physical roots of this equation. The first, $\theta = 0^\circ$, corresponds to the case where the centre of mass of the object is aligned vertically *below* the pivot point. The second, $\theta = 180^\circ$, corresponds to the case where the centre of mass is aligned vertically *above* the pivot point. Of course, the former root is far more important than the latter, since the former root corresponds to a *stable equilibrium*, whereas the latter corresponds to an *unstable equilibrium*. We recall, from Sect. 5.7, that when a system is slightly disturbed from a stable equilibrium then the forces and torques which act upon it tend to return it to this equilibrium, and *vice versa* for an unstable equilibrium. The easiest way to distinguish between stable and unstable equilibria, in the present case, is to evaluate the gravitational potential energy of the system. The potential energy of the object shown in Fig. 90, calculated using the height of the pivot as the reference height, is simply

$$U = -M\,g\,h = -M\,g\,r\,\cos\theta. \tag{10.14}$$

(Note that the gravitational potential energy of an extended object can be calculated by imagining that all of the mass of the object is concentrated at its centre of mass.) It can be seen that $\theta = 0^\circ$ corresponds to a minimum of this potential, whereas $\theta = 180^\circ$ corresponds to a maximum. This is in accordance with Sect. 5.7, where it was demonstrated that whenever an object moves in a conservative force-field (such as a gravitational field), the stable equilibrium points correspond to *minima* of the potential energy associated with this field, whereas the unstable equilibrium points correspond to *maxima*.

10.4 Rods and cables

Consider a uniform rod of mass M and length l which is suspended horizontally via two vertical cables. Let the points of attachment of the two cables be located distances x_1 and x_2 from one of the ends of the rod, labeled A. It is assumed that $x_2 > x_1$. See Fig. 91. What are the tensions, T_1 and T_2, in the cables?

Let us first locate the centre of mass of the rod, which is situated at the rod's mid-point, a distance $l/2$ from reference point A (see Fig. 91). There are three forces acting on the rod: the gravitational force, $M\,g$, and the two tension forces, T_1 and T_2. Each of these forces is directed vertically. Thus, the condition that zero net force acts on the system reduces to the condition that the net vertical force is zero, which yields

$$T_1 + T_2 - M\,g = 0. \tag{10.15}$$

Consider the torques exerted by the three above-mentioned forces about point A. Each of these torques attempts to twist the rod about an axis perpendicular to the plane of the diagram. Hence, the condition that zero net torque acts on the system reduces to the condition that the net torque at point A, about an axis perpendicular to the plane of the diagram, is zero. The contribution of each force to this torque is simply the product of the magnitude of the force and the length of the associated lever arm. In each case, the length of the lever arm is equivalent to the distance of the point of action of the force from A, measured along the length of the rod. Hence, setting the net torque to zero, we obtain

$$x_1\,T_1 + x_2\,T_2 - \frac{l}{2}\,M\,g = 0. \tag{10.16}$$

Note that the torque associated with the gravitational force, $M\,g$, has a minus sign in front, because this torque obviously attempts to twist the rod in the opposite direction to the torques associated with the tensions in the cables.

The previous two equations can be solved to give

$$T_1 = \left(\frac{x_2 - l/2}{x_2 - x_1}\right) M\,g, \tag{10.17}$$

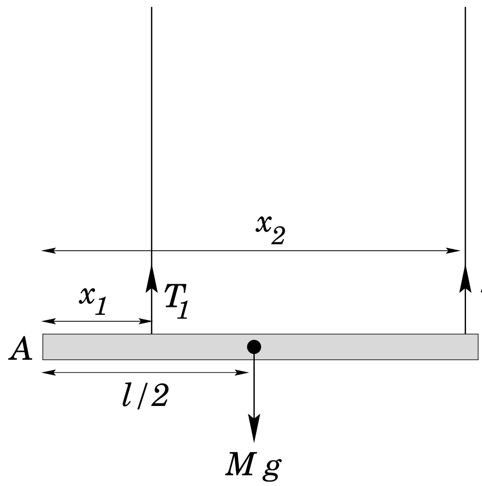


Figure 91: *A horizontal rod suspended by two vertical cables.*

$$T_2 \;=\; \left(\frac{l/2 - x_1}{x_2 - x_1}\right) M\,g. \tag{10.18}$$

Recall that tensions in flexible cables can never be negative, since this would imply that the cables in question were being compressed. Of course, when cables are compressed they simply collapse. It is clear, from the above expressions, that in order for the tensions T_1 and T_2 to remain positive (given that $x_2 > x_1$), the following conditions must be satisfied:

$$x_1 \;<\; \frac{l}{2}, \tag{10.19}$$

$$x_2 \;>\; \frac{l}{2}. \tag{10.20}$$

In other words, the attachment points of the two cables must *straddle* the centre of mass of the rod.

Consider a uniform rod of mass M and length l which is free to rotate in the vertical plane about a fixed pivot attached to one of its ends. The other end of the rod is attached to a fixed cable. We can imagine that both the pivot and the cable are anchored in the same vertical wall. See Fig. 92. Suppose that the rod is level, and that the cable subtends an angle θ with the horizontal. Assuming that the rod is in equilibrium, what is the magnitude of the tension, T, in the cable, and what is the direction and magnitude of the reaction, R, at the pivot?

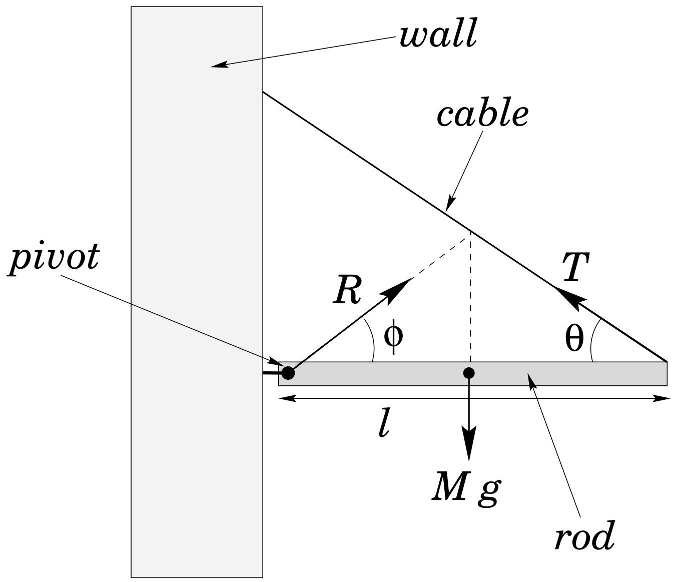


Figure 92: *A rod suspended by a fixed pivot and a cable.*

As usual, the centre of mass of the rod lies at its mid-point. There are three forces acting on the rod: the reaction, R; the weight, $M\,g$; and the tension, T. The reaction acts at the pivot. Let ϕ be the angle subtended by the reaction with the horizontal, as shown in Fig. 92. The weight acts at the centre of mass of the rod, and is directed vertically downwards. Finally, the tension acts at the end of the rod, and is directed along the cable.

Resolving horizontally, and setting the net horizontal force acting on the rod to zero, we obtain

$$R\,\cos\phi - T\,\cos\theta = 0. \tag{10.21}$$

Likewise, resolving vertically, and setting the net vertical force acting on the rod to zero, we obtain

$$R\,\sin\phi + T\,\sin\theta - M\,g = 0. \tag{10.22}$$

The above constraints are sufficient to ensure that zero net force acts on the rod.

Let us evaluate the net torque acting at the pivot point (about an axis perpendicular to the plane of the diagram). The reaction, R, does not contribute to this torque, since it acts at the pivot point. The length of the lever arm associated with the weight, $M\,g$, is $l/2$. Simple trigonometry reveals that the length of the lever arm associated with the tension, T, is $l\,\sin\theta$. Hence, setting the net torque about the pivot point to zero, we obtain

$$M\,g\,\frac{l}{2} - T\,l\,\sin\theta = 0. \tag{10.23}$$

Note that there is a minus sign in front of the second torque, since this torque clearly attempts to twist the rod in the opposite sense to the first.

Equations (10.21) and (10.22) can be solved to give

$$T = \frac{\cos\phi}{\sin(\theta+\phi)}\,M\,g, \tag{10.24}$$

$$R = \frac{\cos\theta}{\sin(\theta+\phi)}\,M\,g. \tag{10.25}$$

Substituting Eq. (10.24) into Eq. (10.23), we obtain

$$\sin(\theta+\phi) = 2\,\sin\theta\,\cos\phi. \tag{10.26}$$

The physical solution of this equation is $\phi = \theta$ (recall that $\sin 2\,\theta = 2\,\sin\theta\,\cos\theta$), which determines the direction of the reaction at the pivot. Finally, Eqs. (10.24) and (10.25) yield

$$T = R = \frac{M\,g}{2\,\sin\theta}, \tag{10.27}$$

which determines both the magnitude of the tension in the cable and that of the reaction at the pivot.

One important point to note about the above solution is that if $\phi = \theta$ then the lines of action of the three forces—R, $M\,g$, and T—intersect at the same point, as shown in Fig. 92. This is an illustration of a general rule. Namely, whenever a rigid body is in equilibrium under the action of *three* forces, then these forces are either *mutually parallel*, as shown in Fig. 91, or their lines of action pass through the *same point*, as shown in Fig. 92.

10.5 Ladders and walls

Suppose that a ladder of length l and negligible mass is leaning against a vertical wall, making an angle θ with the horizontal. A workman of mass M climbs

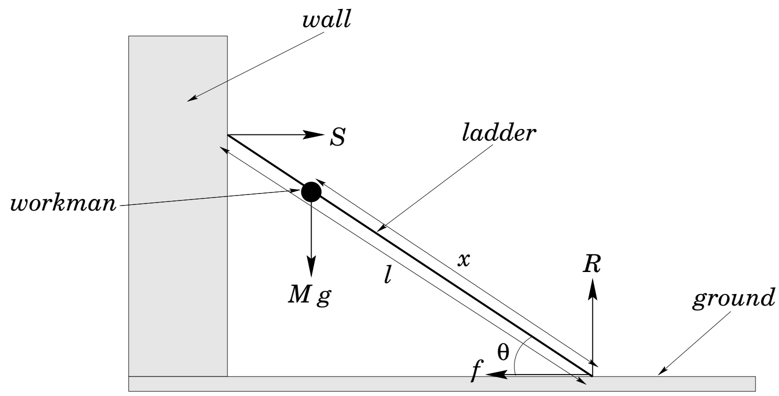


Figure 93: *A ladder leaning against a vertical wall.*

a distance x along the ladder, measured from the bottom. See Fig. 93. Suppose that the wall is completely frictionless, but that the ground possesses a coefficient of static friction μ. How far up the ladder can the workman climb before it slips along the ground? Is it possible for the workman to climb to the top of the ladder without any slippage occurring?

There are four forces acting on the ladder: the weight, $M\,g$, of the workman; the reaction, S, at the wall; the reaction, R, at the ground; and the frictional force, f, due to the ground. The weight acts at the position of the workman, and is directed vertically downwards. The reaction, S, acts at the top of the ladder, and is directed horizontally (*i.e.*, normal to the surface of the wall). The reaction, R, acts at the bottom of the ladder, and is directed vertically upwards (*i.e.*, normal to the ground). Finally, the frictional force, f, also acts at the bottom of the ladder, and is directed horizontally.

Resolving horizontally, and setting the net horizontal force acting on the ladder to zero, we obtain

$$S - f = 0. \tag{10.28}$$

Resolving vertically, and setting the net vertically force acting on the ladder to zero, we obtain

$$R - M\,g = 0. \tag{10.29}$$

Evaluating the torque acting about the point where the ladder touches the ground, we note that only the forces $M\,g$ and S contribute. The lever arm associated with the force $M\,g$ is $x\,\cos\theta$. The lever arm associated with the force S is $l\,\sin\theta$. Furthermore, the torques associated with these two forces act in opposite directions. Hence, setting the net torque about the bottom of the ladder to zero, we obtain

$$M\,g\,x\,\cos\theta - S\,l\,\sin\theta = 0. \tag{10.30}$$

The above three equations can be solved to give

$$R = M\,g, \tag{10.31}$$

and

$$f = S = \frac{x}{l\,\tan\theta}\,M\,g. \tag{10.32}$$

Now, the condition for the ladder not to slip with respect to the ground is

$$f < \mu\,R. \tag{10.33}$$

This condition reduces to

$$x < l\,\mu\,\tan\theta. \tag{10.34}$$

Thus, the furthest distance that the workman can climb along the ladder before it slips is

$$x_{max} = l\,\mu\,\tan\theta. \tag{10.35}$$

Note that if $\tan\theta > 1/\mu$ then the workman can climb all the way along the ladder without any slippage occurring. This result suggests that ladders leaning against walls are less likely to slip when they are almost vertical (*i.e.*, when $\theta \rightarrow 90^\circ$).

10.6 Jointed rods

Suppose that three identical uniform rods of mass M and length l are joined together to form an equilateral triangle, and are then suspended from a cable, as shown in Fig. 94. What is the tension in the cable, and what are the reactions at the joints?

Let X_1, X_2, and X_3 be the horizontal reactions at the three joints, and let Y_1, Y_2, and Y_3 be the corresponding vertical reactions, as shown in Fig. 94. In drawing this diagram, we have made use of the fact that the rods exert equal and opposite

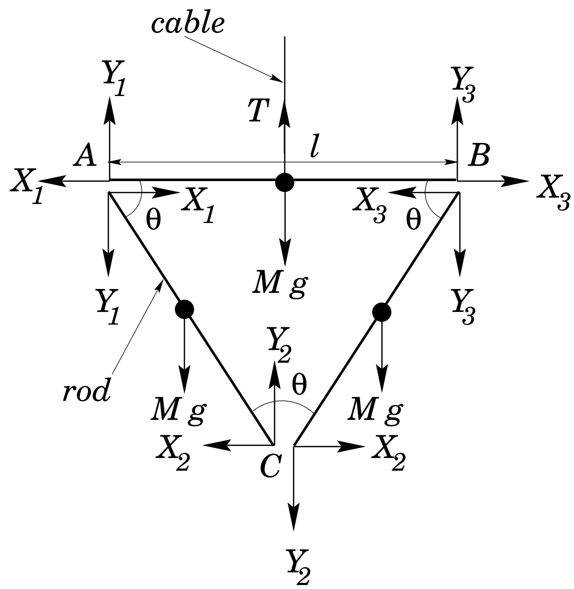


Figure 94: *Three identical jointed rods.*

reactions on one another, in accordance with Newton's third law. Let T be the tension in the cable.

Setting the horizontal and vertical forces acting on rod AB to zero, we obtain

$$X_1 - X_3 = 0, \tag{10.36}$$

$$T + Y_1 + Y_3 - M\,g = 0, \tag{10.37}$$

respectively. Setting the horizontal and vertical forces acting on rod AC to zero, we obtain

$$X_2 - X_1 = 0, \tag{10.38}$$

$$Y_2 - Y_1 - M\,g = 0, \tag{10.39}$$

respectively. Finally, setting the horizontal and vertical forces acting on rod BC to zero, we obtain

$$X_3 - X_2 = 0, \tag{10.40}$$

$$-Y_2 - Y_3 - M\,g = 0, \tag{10.41}$$

respectively. Incidentally, it is clear, from symmetry, that $X_1 = X_3$ and $Y_1 = Y_3$. Thus, the above equations can be solved to give

$$T = 3\,M\,g, \tag{10.42}$$

$$Y_2 = 0, \tag{10.43}$$

$$X_1 = X_2 = X_3 = X, \tag{10.44}$$

$$Y_1 = Y_3 = -M\,g. \tag{10.45}$$

There now remains only one unknown, X.

Now, it is clear, from symmetry, that there is zero net torque acting on rod AB. Let us evaluate the torque acting on rod AC about point A. (By symmetry, this is the same as the torque acting on rod BC about point B). The two forces which contribute to this torque are the weight, $M\,g$, and the reaction $X_2 = X$. (Recall that the reaction Y_2 is zero). The lever arms associated with these two torques (which act in the same direction) are $(l/2)\,\cos\theta$ and $l\,\sin\theta$, respectively. Thus, setting the net torque to zero, we obtain

$$M\,g\,(l/2)\,\cos\theta + X\,l\,\sin\theta = 0, \tag{10.46}$$

which yields

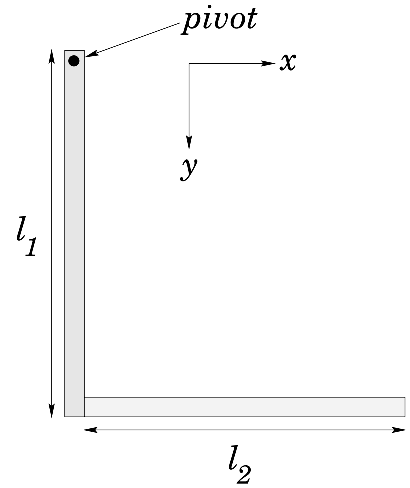
$$X = -\frac{M\,g}{2\,\tan\theta} = -\frac{M\,g}{2\,\sqrt{3}}, \tag{10.47}$$

since $\theta = 60^\circ$, and $\tan 60^\circ = \sqrt{3}$. We have now fully determined the tension in the cable, and all the reactions at the joints.

Worked example 10.1: Equilibrium of two rods

Question: Suppose that two uniform rods (of negligible thickness) are welded together at right-angles, as shown in the diagram below. Let the first rod be of mass $m_1 = 5.2\,\mathrm{kg}$ and length $l_1 = 1.3\,\mathrm{m}$. Let the second rod be of mass $m_2 = 3.4\,\mathrm{kg}$ and length $l_2 = 0.7\,\mathrm{m}$. Suppose that the system is suspended from a pivot point located at the free end of the first rod, and then allowed to reach a stable equilibrium state. What angle θ does the first rod subtend with the downward vertical in this state?

Answer: Let us adopt a coordinate system in which the x-axis runs parallel to the second rod, whereas the y-axis runs parallel to the first. Let the origin of our



coordinate system correspond to the pivot point. The centre of mass of the first rod is situated at its mid-point, whose coordinates are

$$(x_1, y_1) = (0, l_1/2).$$

Likewise, the centre of mass of the second rod is situated at its mid-point, whose coordinates are

$$(x_2, y_2) = (l_2/2, l_1).$$

It follows that the coordinates of the centre of mass of the whole system are given by

$$x_{cm} = \frac{m_1\, x_1 + m_2\, x_2}{m_1 + m_2} = \frac{1}{2}\frac{m_2\, l_2}{m_1 + m_2} = \frac{3.4 \times 0.7}{2 \times 8.6} = 0.138\,\mathrm{m},$$

and

$$y_{cm} = \frac{m_1\, y_1 + m_2\, y_2}{m_1 + m_2} = \frac{m_1\, l_1/2 + m_2\, l_1}{m_1 + m_2} = \frac{5.2 \times 1.3/2 + 3.4 \times 1.3}{8.6} = 0.907\,\mathrm{m}.$$

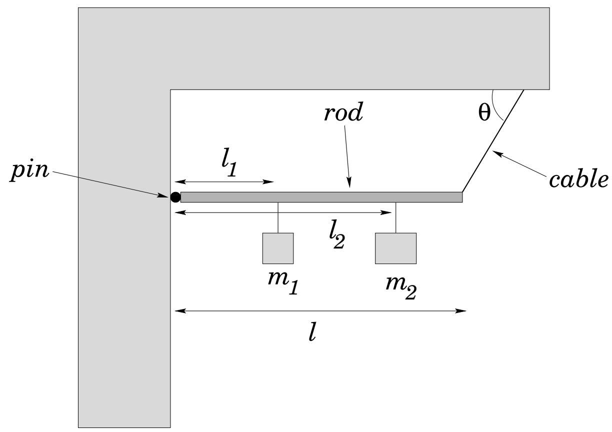
The angle θ subtended between the line joining the pivot point and the overall centre of mass, and the first rod is simply

$$\theta = \tan^{-1}\left(\frac{x_{cm}}{y_{cm}}\right) = \tan^{-1} 0.152 = 8.65^\circ.$$

When the system reaches a stable equilibrium state then its centre of mass is aligned directly below the pivot point. This implies that the first rod subtends an angle $\theta = 8.65^\circ$ with the downward vertical.

Worked example 10.2: Rod supported by a cable

Question: A uniform rod of mass $m = 15\,\mathrm{kg}$ and length $l = 3\,\mathrm{m}$ is supported in a horizontal position by a pin and a cable, as shown in the figure below. Masses $m_1 = 36\,\mathrm{kg}$ and $m_2 = 24\,\mathrm{kg}$ are suspended from the rod at positions $l_1 = 0.5\,\mathrm{m}$ and $l_2 = 2.3\,\mathrm{m}$. The angle θ is 40°. What is the tension T in the cable?



Answer: Consider the torque acting on the rod about the pin. Note that the reaction at the pin makes no contribution to this torque (since the length of the associated lever arm is zero). The torque due to the weight of the rod is $m\,g\,l/2$ (*i.e.*, the weight times the length of the lever arm). Note that the weight of the rod acts at its centre of mass, which is located at the rod's mid-point. The torque due to the weight of the first mass is $m_1\,g\,l_1$. The torque due to the weight of the second mass is $m_2\,g\,l_2$. Finally, the torque due to the tension in the cable is $-T\,l\,\sin\theta$ (this torque is negative since it twists the rod in the opposite sense to the other three torques). Hence, setting the net torque to zero, we obtain

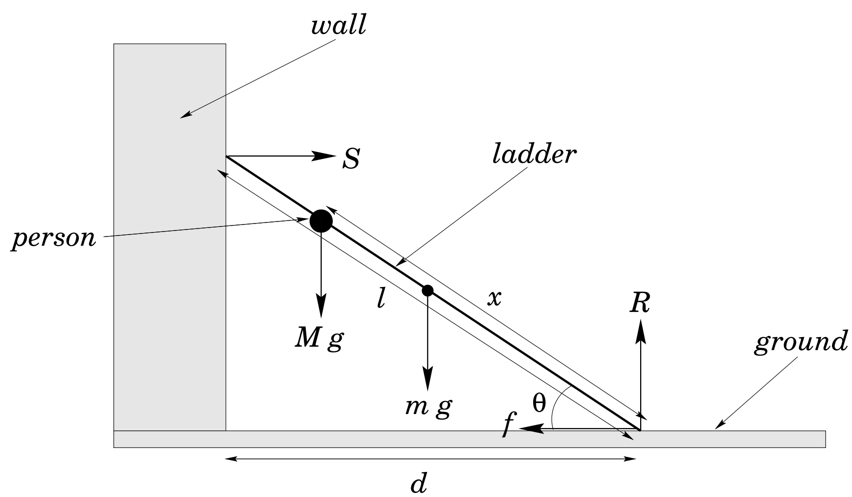
$$m\,g\,\frac{l}{2} + m_1\,g\,l_1 + m_2\,g\,l_2 - T\,l\,\sin\theta = 0,$$

or

$$\begin{aligned} T &= \frac{[m/2 + m_1\,(l_1/l) + m_2\,(l_2/l)]\,g}{\sin\theta} \\ &= \frac{[0.5\times 15 + 36\times(0.5/3) + 24\times(2.3/3)]\times 9.81}{\sin 40^\circ} \\ &= 486.84\,\mathrm{N}. \end{aligned}$$

Worked example 10.3: Leaning ladder

Question: A uniform ladder of mass $m = 40\,\text{kg}$ and length $l = 10\,\text{m}$ is leaned against a smooth vertical wall. A person of mass $M = 80\,\text{kg}$ stands on the ladder a distance $x = 7\,\text{m}$ from the bottom, as measured along the ladder. The foot of the ladder is $d = 1.2\,\text{m}$ from the bottom of the wall. What is the force exerted by the wall on the ladder? What is the normal force exerted by the floor on the ladder?



Answer: The angle θ subtended by the ladder with the ground satisfies

$$\theta = \cos^{-1}(d/l) = \cos^{-1}(1.2/10) = 83.11^\circ.$$

Let S be the normal reaction at the wall, let R be the normal reaction at the ground, and let f be the frictional force exerted by the ground on the ladder, as shown in the diagram. Consider the torque acting on the ladder about the point where it meets the ground. Only three forces contribute to this torque: the weight, $m\,g$, of the ladder, which acts half-way along the ladder; the weight, $M\,g$, of the person, which acts a distance x along the ladder; and the reaction, S, at the wall, which acts at the top of the ladder. The lever arms associated with these three forces are $(l/2)\cos\theta$, $x\cos\theta$, and $l\sin\theta$, respectively. Note that the reaction force acts to twist the ladder in the opposite sense to the two weights. Hence, setting the net torque to zero, we obtain

$$m\,g\,\frac{l}{2}\cos\theta + M\,g\,x\cos\theta - S\,l\sin\theta = 0,$$

which yields

$$S = \frac{(m\,g/2 + M\,g\,x/l)}{\tan\theta} = \frac{(0.5\times 40\times 9.81 + 80\times 9.81\times 7/10)}{\tan 83.11^\circ} = 90.09\,\mathrm{N}.$$

The condition that zero net vertical force acts on the ladder yields

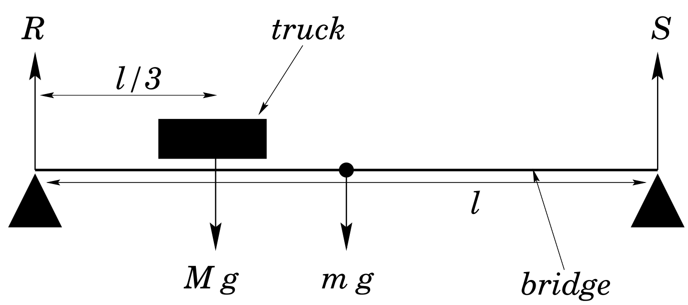
$$R - m\,g - M\,g = 0.$$

Hence,

$$R = (m + M)\,g = (40 + 80)\times 9.81 = 1177.2\,\mathrm{N}.$$

Worked example 10.4: Truck crossing a bridge

Question: A truck of mass $M = 5000\,\mathrm{kg}$ is crossing a uniform horizontal bridge of mass $m = 1000\,\mathrm{kg}$ and length $l = 100\,\mathrm{m}$. The bridge is supported at its two end-points. What are the reactions at these supports when the truck is one third of the way across the bridge?



Answer: Let R and S be the reactions at the bridge supports. Here, R is the reaction at the support closest to the truck. Setting the net vertical force acting on the bridge to zero, we obtain

$$R + S - M\,g - m\,g = 0.$$

Setting the torque acting on the bridge about the left-most support to zero, we get

$$M\,g\,l/3 + m\,g\,l/2 - S\,l = 0.$$

Here, we have made use of the fact that centre of mass of the bridge lies at its mid-point. It follows from the above two equations that

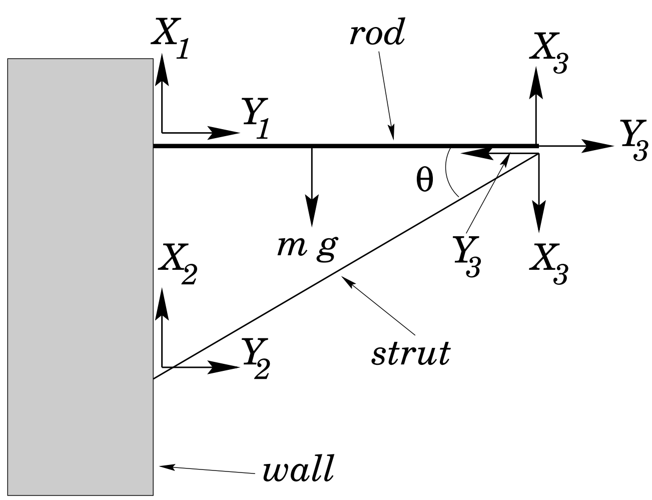
$$S = M\,g/3 + m\,g/2 = 5000 \times 9.81/3 + 1000 \times 9.81/2 = 2.13 \times 10^4\,\text{N},$$

and

$$R = M\,g + m\,g - S = (5000 + 1000) \times 9.81 - 2.13 \times 10^4 = 3.76 \times 10^4\,\text{N}.$$

Worked example 10.5: Rod supported by a strut

Question: A uniform horizontal rod of mass $m = 15\,\text{kg}$ is attached to a vertical wall at one end, and is supported, from below, by a light rigid strut at the other. The strut is attached to the rod at one end, and the wall at the other, and subtends an angle of $\theta = 30^\circ$ with the rod. Find the horizontal and vertical reactions at the point where the strut is attached to the rod, and the points where the rod and the strut are attached to the wall.



Answer: Let us call the vertical reactions at the joints X_1, X_2, and X_3. Let the corresponding horizontal reactions be Y_1, Y_2, and Y_3. See the diagram. Here, we have made use of the fact that the strut and the rod exert equal and opposite reactions on one another, in accordance with Newton's third law. Setting the net vertical force on the rod to zero yields

$$X_1 + X_3 - m\,g = 0.$$

Setting the net horizontal force on the rod to zero gives

$$Y_1 + Y_3 = 0.$$

Setting the net vertical force on the strut to zero yields

$$X_2 - X_3 = 0.$$

Finally, setting the net horizontal force on the strut to zero yields

$$Y_2 - Y_3 = 0.$$

The above equations can be solved to give

$$-Y_1 = Y_2 = Y_3 = Y,$$

and

$$X_2 = X_3 = X,$$

with

$$X_1 = m\,g - X.$$

There now remain only two unknowns, X and Y.

Setting the net torque acting on the rod about the point where it is connected to the wall to zero, we obtain

$$m\,g\,l/2 - X_3\,l = 0,$$

where l is the length of the rod. Here, we have used the fact that the centre of gravity of the rod lies at its mid-point. The above equation implies that

$$X_3 = X = m\,g/2 = 15 \times 9.81/2 = 73.58\,\text{N}.$$

We also have $X_1 = m\,g - X = 73.58\,\text{N}$. Setting the net torque acting on the strut about the point where it is connected to the wall to zero, we find

$$Y_3\,h\,\sin\theta - X_3\,h\,\cos\theta = 0,$$

where h is the length of the strut. Thus,

$$Y_3 = Y = \frac{X}{\tan\theta} = \frac{73.58}{\tan 30^\circ} = 127.44\,\text{N}.$$

In summary, the vertical reactions are $X_1 = X_2 = X_3 = 73.58\,\text{N}$, and the horizontal reactions are $-Y_1 = Y_2 = Y_3 = 127.44\,\text{N}$.

11 Oscillatory motion

11.1 Introduction

We have seen previously (for instance, in Sect. 10.3) that when systems are perturbed from a *stable* equilibrium state they experience a *restoring force* which acts to return them to that state. In many cases of interest, the magnitude of the restoring force is directly proportional to the displacement from equilibrium. In this section, we shall investigate the motion of systems subject to such a force.

11.2 Simple harmonic motion

Let us reexamine the problem of a mass on a spring (see Sect. 5.6). Consider a mass m which slides over a horizontal frictionless surface. Suppose that the mass is attached to a light horizontal spring whose other end is anchored to an immovable object. See Fig. 42. Let x be the extension of the spring: *i.e.*, the difference between the spring's actual length and its unstretched length. Obviously, x can also be used as a coordinate to determine the horizontal displacement of the mass.

The equilibrium state of the system corresponds to the situation where the mass is at rest, and the spring is unextended (*i.e.*, $x = 0$). In this state, zero net force acts on the mass, so there is no reason for it to start to move. If the system is perturbed from this equilibrium state (*i.e.*, if the mass is moved, so that the spring becomes extended) then the mass experiences a *restoring force* given by Hooke's law:

$$f = -k\,x. \tag{11.1}$$

Here, $k > 0$ is the *force constant* of the spring. The negative sign indicates that f is indeed a restoring force. Note that the magnitude of the restoring force is *directly proportional* to the displacement of the system from equilibrium (*i.e.*, $f \propto x$). Of course, Hooke's law only holds for *small* spring extensions. Hence, the displacement from equilibrium cannot be made too large. The motion of this system is representative of the motion of a wide range of systems when they are *slightly* disturbed from a stable equilibrium state.

Newton's second law gives following equation of motion for the system:

$$m\,\ddot{x} = -k\,x. \tag{11.2}$$

This *differential equation* is known as the *simple harmonic equation*, and its solution has been known for centuries. In fact, the solution is

$$x = a\,\cos(\omega\,t - \phi), \tag{11.3}$$

where a, ω, and ϕ are constants. We can demonstrate that Eq. (11.3) is indeed a solution of Eq. (11.2) by direct substitution. Substituting Eq. (11.3) into Eq. (11.2), and recalling from calculus that $d(\cos\theta)/d\theta = -\sin\theta$ and $d(\sin\theta)/d\theta = \cos\theta$, we obtain

$$-\,m\,\omega^2\,a\,\cos(\omega\,t - \phi) = -k\,a\,\cos(\omega\,t - \phi). \tag{11.4}$$

It follows that Eq. (11.3) is the correct solution provided

$$\omega = \sqrt{\frac{k}{m}}. \tag{11.5}$$

Figure 95 shows a graph of x versus t obtained from Eq. (11.3). The type of motion shown here is called *simple harmonic motion*. It can be seen that the displacement x *oscillates* between $x = -a$ and $x = +a$. Here, a is termed the *amplitude* of the oscillation. Moreover, the motion is *periodic* in time (*i.e.*, it repeats exactly after a certain time period has elapsed). In fact, the *period* is

$$T = \frac{2\,\pi}{\omega}. \tag{11.6}$$

This result is easily obtained from Eq. (11.3) by noting that $\cos\theta$ is a periodic function of θ with period $2\,\pi$. The frequency of the motion (*i.e.*, the number of oscillations completed per second) is

$$f = \frac{1}{T} = \frac{\omega}{2\,\pi}. \tag{11.7}$$

$\omega t - \phi$	0°	90°	180°	270°
x	$+a$	0	$-a$	0
$\dot{x}$	0	$-\omega a$	0	$+\omega a$
$\ddot{x}$	$-\omega^2 a$	0	$+\omega^2 a$	0

Table 4: *Simple harmonic motion.*

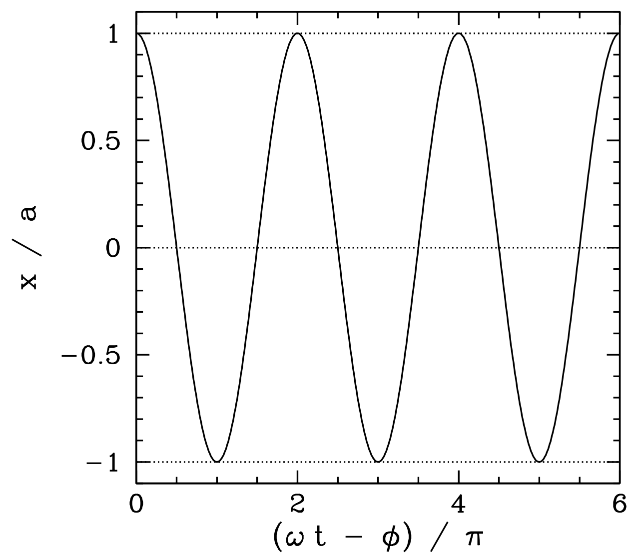


Figure 95: *Simple harmonic motion.*

It can be seen that ω is the motion's *angular frequency* (*i.e.*, the frequency f converted into radians per second). Finally, the *phase angle* ϕ determines the times at which the oscillation attains its maximum amplitude, $x = a$: in fact,

$$t_{max} = T\left(n + \frac{\phi}{2\pi}\right). \tag{11.8}$$

Here, n is an arbitrary integer.

Table 4 lists the displacement, velocity, and acceleration of the mass at various phases of the simple harmonic cycle. The information contained in this table can easily be derived from the simple harmonic equation, Eq. (11.3). Note that all of the non-zero values shown in this table represent either the maximum or the minimum value taken by the quantity in question during the oscillation cycle.

We have seen that when a mass on a spring is disturbed from equilibrium it executes *simple harmonic motion* about its equilibrium state. In physical terms, if the initial displacement is positive ($x > 0$) then the restoring force *overcompensates*, and sends the system past the equilibrium state ($x = 0$) to negative displacement states ($x < 0$). The restoring force again overcompensates, and sends the system back through $x = 0$ to positive displacement states. The motion then repeats itself *ad infinitum*. The frequency of the oscillation is determined by the spring stiffness, k, and the system inertia, m, via Eq. (11.5). In contrast, the amplitude and phase angle of the oscillation are determined by the *initial conditions*. Suppose that the instantaneous displacement and velocity of the mass at $t = 0$ are x_0 and v_0, respectively. It follows from Eq. (11.3) that

$$x_0 = x(t = 0) = a \cos\phi, \tag{11.9}$$

$$v_0 = \dot{x}(t = 0) = a\,\omega\,\sin\phi. \tag{11.10}$$

Here, use has been made of the well-known identities $\cos(-\theta) = \cos\theta$ and $\sin(-\theta) = -\sin\theta$. Hence, we obtain

$$a = \sqrt{x_0^2 + (v_0/\omega)^2}, \tag{11.11}$$

and

$$\phi = \tan^{-1}\left(\frac{v_0}{\omega\,x_0}\right), \tag{11.12}$$

since $\sin^2\theta + \cos^2\theta = 1$ and $\tan\theta = \sin\theta/\cos\theta$.

The kinetic energy of the system is written

$$K = \frac{1}{2}\,m\,\dot{x}^2 = \frac{m\,a^2\,\omega^2\,\sin^2(\omega\,t - \phi)}{2}. \tag{11.13}$$

Recall, from Sect. 5.6, that the potential energy takes the form

$$U = \frac{1}{2}\,k\,x^2 = \frac{k\,a^2\,\cos^2(\omega\,t - \phi)}{2}. \tag{11.14}$$

Hence, the total energy can be written

$$E = K + U = \frac{a^2\,k}{2}, \tag{11.15}$$

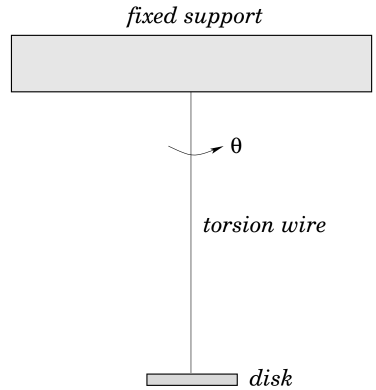


Figure 96: *A torsion pendulum.*

since $m\,\omega^2 = k$ and $\sin^2\theta + \cos^2\theta = 1$. Note that the total energy is a *constant of the motion*, as expected for an isolated system. Moreover, the energy is proportional to the *amplitude squared* of the motion. It is clear, from the above expressions, that simple harmonic motion is characterized by a constant backward and forward flow of energy between kinetic and potential components. The kinetic energy attains its maximum value, and the potential energy attains it minimum value, when the displacement is zero (*i.e.*, when $x = 0$). Likewise, the potential energy attains its maximum value, and the kinetic energy attains its minimum value, when the displacement is maximal (*i.e.*, when $x = \pm a$). Note that the minimum value of K is *zero*, since the system is instantaneously at rest when the displacement is maximal.

11.3 The torsion pendulum

Consider a disk suspended from a torsion wire attached to its centre. See Fig. 96. This setup is known as a *torsion pendulum*. A torsion wire is essentially inextensible, but is free to *twist* about its axis. Of course, as the wire twists it also causes the disk attached to it to *rotate* in the horizontal plane. Let θ be the angle of rotation of the disk, and let $\theta = 0$ correspond to the case in which the wire is untwisted.

Any twisting of the wire is inevitably associated with mechanical deformation. The wire resists such deformation by developing a *restoring torque*, τ, which acts to restore the wire to its untwisted state. For relatively small angles of twist, the magnitude of this torque is directly proportional to the twist angle. Hence, we can write

$$\tau = -k\,\theta, \tag{11.16}$$

where $k > 0$ is the *torque constant* of the wire. The above equation is essentially a torsional equivalent to Hooke's law. The rotational equation of motion of the system is written

$$I\,\ddot{\theta} = \tau, \tag{11.17}$$

where I is the moment of inertia of the disk (about a perpendicular axis through its centre). The moment of inertia of the wire is assumed to be negligible. Combining the previous two equations, we obtain

$$I\,\ddot{\theta} = -k\,\theta. \tag{11.18}$$

Equation (11.18) is clearly a simple harmonic equation [*cf.*, Eq. (11.2)]. Hence, we can immediately write the standard solution [*cf.*, Eq. (11.3)]

$$\theta = a\,\cos(\omega\,t - \phi), \tag{11.19}$$

where [*cf.*, Eq. (11.5)]

$$\omega = \sqrt{\frac{k}{I}}. \tag{11.20}$$

We conclude that when a torsion pendulum is perturbed from its equilibrium state (*i.e.*, $\theta = 0$), it executes torsional oscillations about this state at a fixed frequency, ω, which depends only on the torque constant of the wire and the moment of inertia of the disk. Note, in particular, that the frequency is independent of the amplitude of the oscillation [provided θ remains small enough that Eq. (11.16) still applies]. Torsion pendulums are often used for time-keeping purposes. For instance, the balance wheel in a mechanical wristwatch is a torsion pendulum in which the restoring torque is provided by a coiled spring.

11.4 The simple pendulum

Consider a mass m suspended from a light inextensible string of length l, such that the mass is free to swing from side to side in a vertical plane, as shown in

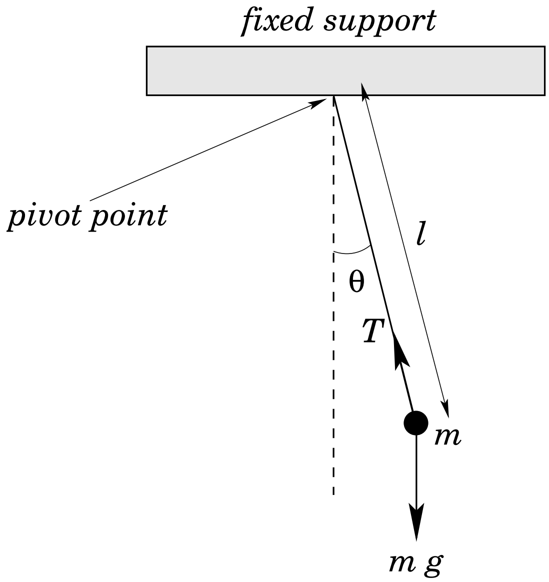


Figure 97: *A simple pendulum.*

Fig. 97. This setup is known as a *simple pendulum*. Let θ be the angle subtended between the string and the downward vertical. Obviously, the equilibrium state of the simple pendulum corresponds to the situation in which the mass is stationary and hanging vertically down (*i.e.*, $\theta = 0$). The angular equation of motion of the pendulum is simply

$$I\,\ddot{\theta} = \tau, \tag{11.21}$$

where I is the moment of inertia of the mass, and τ is the torque acting on the system. For the case in hand, given that the mass is essentially a point particle, and is situated a distance l from the axis of rotation (*i.e.*, the pivot point), it is easily seen that $I = m\,l^2$.

The two forces acting on the mass are the downward gravitational force, $m\,g$, and the tension, T, in the string. Note, however, that the tension makes no contribution to the torque, since its line of action clearly passes through the pivot point. From simple trigonometry, the line of action of the gravitational force passes a distance $l\,\sin\theta$ from the pivot point. Hence, the magnitude of the gravitational torque is $m\,g\,l\,\sin\theta$. Moreover, the gravitational torque is a *restoring torque*: *i.e.*, if the mass is displaced slightly from its equilibrium state (*i.e.*, $\theta = 0$) then the gravitational force clearly acts to push the mass back toward that state.

Thus, we can write

$$\tau = -m\,g\,l\,\sin\theta. \tag{11.22}$$

Combining the previous two equations, we obtain the following angular equation of motion of the pendulum:

$$l\,\ddot{\theta} = -g\,\sin\theta. \tag{11.23}$$

Unfortunately, this is *not* the simple harmonic equation. Indeed, the above equation possesses no closed solution which can be expressed in terms of simple functions.

Suppose that we restrict our attention to relatively *small* deviations from the equilibrium state. In other words, suppose that the angle θ is constrained to take fairly small values. We know, from trigonometry, that for $|\theta|$ less than about 6° it is a good approximation to write

$$\sin\theta \simeq \theta. \tag{11.24}$$

Hence, in the *small angle limit*, Eq. (11.23) reduces to

$$l\,\ddot{\theta} = -g\,\theta, \tag{11.25}$$

which *is* in the familiar form of a simple harmonic equation. Comparing with our original simple harmonic equation, Eq. (11.2), and its solution, we conclude that the angular frequency of small amplitude oscillations of a simple pendulum is given by

$$\omega = \sqrt{\frac{g}{l}}. \tag{11.26}$$

In this case, the pendulum frequency is dependent only on the length of the pendulum and the local gravitational acceleration, and is independent of the mass of the pendulum and the amplitude of the pendulum swings (provided that $\sin\theta \simeq \theta$ remains a good approximation). Historically, the simple pendulum was the basis of virtually all accurate time-keeping devices before the advent of electronic clocks. Simple pendulums can also be used to measure local variations in g.

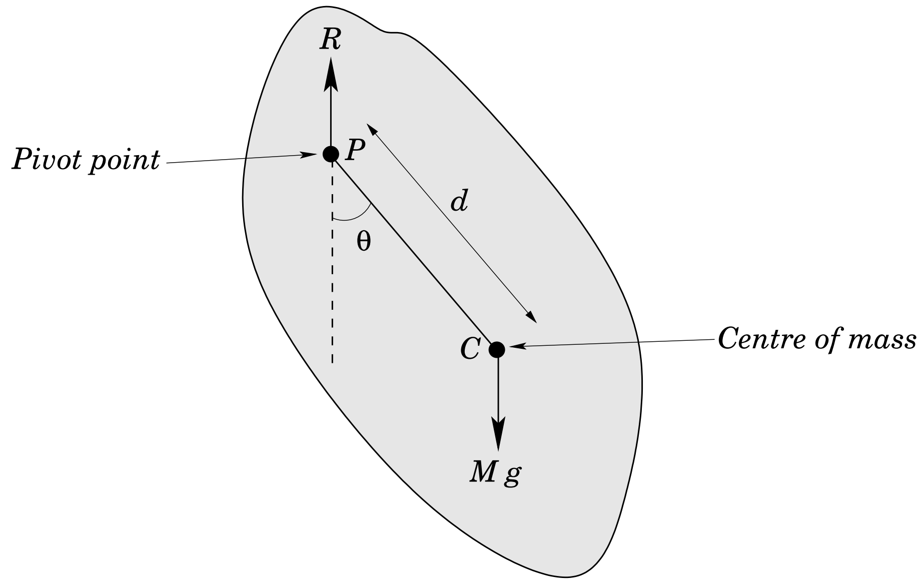


Figure 98: *A compound pendulum.*

11.5 The compound pendulum

Consider an extended body of mass M with a hole drilled though it. Suppose that the body is suspended from a fixed peg, which passes through the hole, such that it is free to swing from side to side, as shown in Fig. 98. This setup is known as a *compound pendulum*.

Let P be the pivot point, and let C be the body's centre of mass, which is located a distance d from the pivot. Let θ be the angle subtended between the downward vertical (which passes through point P) and the line PC. The equilibrium state of the compound pendulum corresponds to the case in which the centre of mass lies vertically below the pivot point: *i.e.*, $\theta = 0$. See Sect. 10.3. The angular equation of motion of the pendulum is simply

$$I\,\ddot{\theta} = \tau, \tag{11.27}$$

where I is the moment of inertia of the body about the pivot point, and τ is the torque. Using similar arguments to those employed for the case of the simple pendulum (recalling that all the weight of the pendulum acts at its centre of mass), we can write

$$\tau = -M\,g\,d\,\sin\theta. \tag{11.28}$$

Note that the reaction, R, at the peg does not contribute to the torque, since its line of action passes through the pivot point. Combining the previous two equations, we obtain the following angular equation of motion of the pendulum:

$$I\,\ddot{\theta} = -M\,g\,d\,\sin\theta. \tag{11.29}$$

Finally, adopting the small angle approximation, $\sin\theta \simeq \theta$, we arrive at the simple harmonic equation:

$$I\,\ddot{\theta} = -M\,g\,d\,\theta. \tag{11.30}$$

It is clear, by analogy with our previous solutions of such equations, that the angular frequency of small amplitude oscillations of a compound pendulum is given by

$$\omega = \sqrt{\frac{M\,g\,d}{I}}. \tag{11.31}$$

It is helpful to define the length

$$L = \frac{I}{M\,d}. \tag{11.32}$$

Equation (11.31) reduces to

$$\omega = \sqrt{\frac{g}{L}}, \tag{11.33}$$

which is identical in form to the corresponding expression for a simple pendulum. We conclude that a compound pendulum behaves like a simple pendulum with *effective length* L.

11.6 Uniform circular motion

Consider an object executing uniform circular motion of radius a. Let us set up a cartesian coordinate system whose origin coincides with the centre of the circle, and which is such that the motion is confined to the x-y plane. As illustrated in Fig. 99, the instantaneous position of the object can be conveniently parameterized in terms of an angle θ.

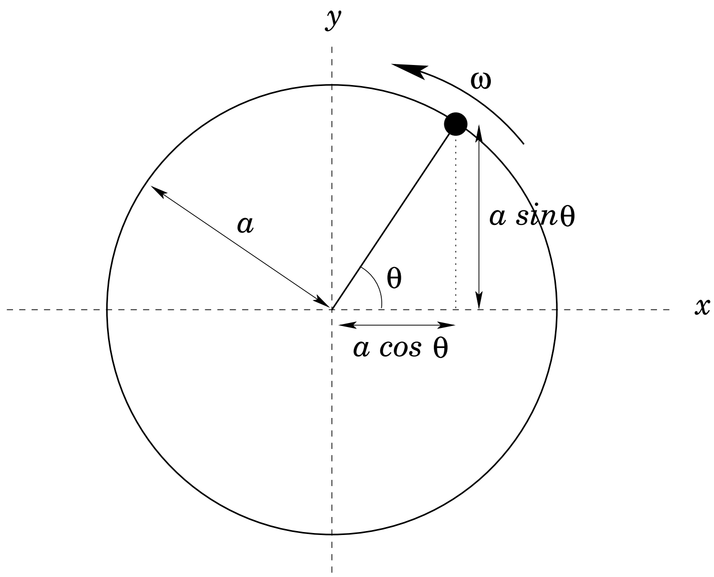


Figure 99: *Uniform circular motion.*

Since the object is executing *uniform* circular motion, we expect the angle θ to increase *linearly* with time. In other words, we can write

$$\theta = \omega\, t, \tag{11.34}$$

where ω is the angular rotation frequency (*i.e.*, the number of radians through which the object rotates per second). Here, it is assumed that $\theta = 0$ at $t = 0$, for the sake of convenience.

From simple trigonometry, the x- and y-coordinates of the object can be written

$$x = a\,\cos\theta, \tag{11.35}$$

$$y = a\,\sin\theta, \tag{11.36}$$

respectively. Hence, combining the previous equations, we obtain

$$x = a\,\cos(\omega\, t), \tag{11.37}$$

$$y = a\,\cos(\omega\, t - \pi/2). \tag{11.38}$$

Here, use has been made of the trigonometric identity $\sin\theta = \cos(\theta - \pi/2)$. A comparison of the above two equations with the standard equation of simple harmonic motion, Eq. (11.3), reveals that our object is executing simple harmonic motion simultaneously along both the x- and the y -axes. Note, however, that these two motions are 90° (*i.e.*, $\pi/2$ radians) *out of phase*. Moreover, the amplitude of the motion equals the radius of the circle. Clearly, there is a close relationship between simple harmonic motion and circular motion.

Worked example 11.1: Piston in steam engine

Question: A piston in a stream engine executes simple harmonic motion. Given that the maximum displacement of the piston from its centre-line is ± 7 cm, and that the mass of the piston is 4 kg, find the maximum velocity of the piston when the steam engine is running at 4000 rev./min. What is the maximum acceleration?

Answer: We are told that the amplitude of the oscillation is $a = 0.07$ m. Moreover, when converted to cycles per second (*i.e.*, hertz), the frequency of the oscillation becomes

$$f = \frac{4000}{60} = 66.6666\,\mathrm{Hz}.$$

Hence, the angular frequency is

$$\omega = 2\pi f = 418.88\,\mathrm{rad./sec}.$$

Consulting Tab. 4, we note that the maximum velocity of an object executing simple harmonic motion is $v_{max} = a\,\omega$. Hence, the maximum velocity is

$$v_{max} = a\,\omega = 0.07 \times 418.88 = 29.32\,\mathrm{m/s}.$$

Likewise, according to Tab. 4, the maximum acceleration is given by

$$a_{max} = a\,\omega^2 = 0.07 \times 418.88 \times 418.88 = 1.228 \times 10^4\,\mathrm{m/s^2}.$$

Worked example 11.2: Block and spring

Question: A block attached to a spring executes simple harmonic motion in a horizontal plane with an amplitude of 0.25 m. At a point 0.15 m away from the equilibrium position, the velocity of the block is 0.75 m/s. What is the period of oscillation of the block?

Answer: The equation of simple harmonic motion is

$$x = a\,\cos(\omega\,t - \phi),$$

where x is the displacement, and a is the amplitude. We are told that $a = 0.25$ m. The velocity of the block is obtained by taking the time derivative of the above expression:

$$\dot{x} = -a\,\omega\,\sin(\omega\,t - \phi).$$

We are told that at $t = 0$ (say), $x = 0.15$ m and $\dot{x} = 0.75$ m/s. Hence,

$$\begin{aligned} 0.15 &= 0.25\,\cos(\phi), \\ 0.75 &= 0.25\,\omega\,\sin(\phi). \end{aligned}$$

The first equation gives $\phi = \cos^{-1}(0.15/0.25) = 53.13^\circ$. The second equation yields

$$\omega = \frac{0.75}{0.25 \times \sin(53.13^\circ)} = 3.75\,\text{rad./s}.$$

Hence, the period of the motion is

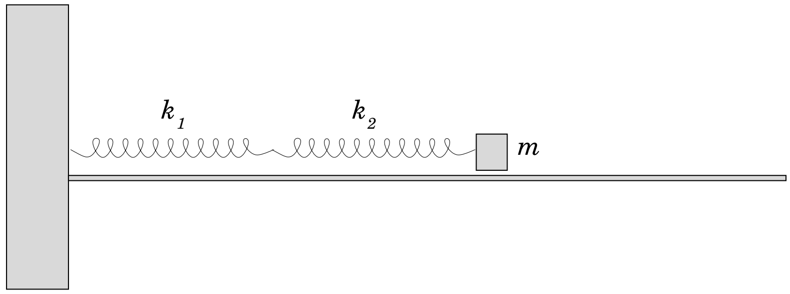
$$T = \frac{2\,\pi}{\omega} = 1.676\,\text{s}.$$

Worked example 11.3: Block and two springs

Question: A block of mass $m = 3$ kg is attached to two springs, as shown below, and slides over a horizontal frictionless surface. Given that the force constants of the two springs are $k_1 = 1200$ N/m and $k_2 = 400$ N/m, find the period of oscillation of the system.

Answer: Let x_1 and x_2 represent the extensions of the first and second springs, respectively. The net displacement x of the mass from its equilibrium position is then given by

$$x = x_1 + x_2.$$



Let $f_1 = k_1 x_1$ and $f_2 = k_2 x_2$ be the magnitudes of the forces exerted by the first and second springs, respectively. Since the springs (presumably) possess negligible inertia, they must exert equal and opposite forces on one another. This implies that $f_1 = f_2$, or

$$k_1 x_1 = k_2 x_2.$$

Finally, if f is the magnitude of the restoring force acting on the mass, then force balance implies that $f = f_1 = f_2$, or

$$f = k_{\rm eff}\, x = k_1 x_1.$$

Here, $k_{\rm eff}$ is the effective force constant of the two springs. The above equations can be combined to give

$$k_{\rm eff} = \frac{k_1 x_1}{x_1 + x_2} = \frac{k_1}{1 + k_1/k_2} = \frac{k_1 k_2}{k_1 + k_2}.$$

Thus, the problem reduces to that of a block of mass $m = 3\,{\rm kg}$ attached to a spring of effective force constant

$$k_{\rm eff} = \frac{k_1 k_2}{k_1 + k_2} = \frac{1200 \times 400}{1200 + 400} = 300\,{\rm N/m}.$$

The angular frequency of oscillation is immediately given by the standard formula

$$\omega = \sqrt{\frac{k_{\rm eff}}{m}} = \sqrt{\frac{300}{3}} = 10\,{\rm rad./s}.$$

Hence, the period of oscillation is

$$T = \frac{2\pi}{\omega} = 0.6283\,{\rm s}.$$

Worked example 11.4: Energy in simple harmonic motion

Question: A block of mass $m = 4$ kg is attached to a spring, and undergoes simple harmonic motion with a period of $T = 0.35$ s. The total energy of the system is $E = 2.5$ J. What is the force constant of the spring? What is the amplitude of the motion?

Answer: The angular frequency of the motion is

$$\omega = \frac{2\,\pi}{T} = \frac{2\,\pi}{0.35} = 17.95\,\text{rad./s.}$$

Now, $\omega = \sqrt{k/m}$ for a mass on a spring. Rearrangement of this formula yields

$$k = m\,\omega^2 = 4 \times 17.95 \times 17.95 = 1289.1\,\text{N/m.}$$

The total energy of a system executing simple harmonic motion is $E = a^2\,k/2$. Rearrangement of this formula gives

$$a = \sqrt{\frac{2\,E}{k}} = \sqrt{\frac{2 \times 2.5}{1289.1}} = 0.06228\,\text{m.}$$

Thus, the force constant is 1289.1 N/m and the amplitude is 0.06228 m.

Worked example 11.5: Gravity on a new planet

Question: Having landed on a newly discovered planet, an astronaut sets up a simple pendulum of length 0.6 m, and finds that it makes 51 complete oscillations in 1 minute. The amplitude of the oscillations is small compared to the length of the pendulum. What is the surface gravitational acceleration on the planet?

Answer: The frequency of the oscillations is

$$f = \frac{51}{60} = 0.85\,\text{Hz.}$$

Hence, the angular frequency is

$$\omega = 2\,\pi\,f = 2 \times \pi \times 1.833 = 5.341\,\text{rad./s.}$$

Now, $\omega = \sqrt{g/l}$ for small amplitude oscillations of a simple pendulum. Rearrangement off this formula gives

$$g = \omega^2\, l = 5.341 \times 5.341 \times 0.6 = 17.11\,\mathrm{m/s^2}.$$

Hence, the surface gravitational acceleration is $17.11\,\mathrm{m/s^2}$.

Worked example 11.6: Oscillating disk

Question: A uniform disk of radius $r = 0.8\,\mathrm{m}$ and mass $M = 3\,\mathrm{kg}$ is freely suspended from a horizontal pivot located a radial distance $d = 0.25\,\mathrm{m}$ from its centre. Find the angular frequency of small amplitude oscillations of the disk.

Answer: The moment of inertia of the disk about a perpendicular axis passing through its centre is $I = (1/2)\, M\, r^2$. From the parallel axis theorem, the moment of inertia of the disk about the pivot point is

$$I' = I + M\, d^2 = \frac{3 \times 0.8 \times 0.8}{2} + 3 \times 0.25 \times 0.25 = 1.1475\,\mathrm{kg\,m^2}.$$

The angular frequency of small amplitude oscillations of a compound pendulum is given by

$$\omega = \sqrt{\frac{M\, g\, d}{I'}} = \sqrt{\frac{3 \times 9.81 \times 0.25}{1.1475}} = 2.532\,\mathrm{rad./s}.$$

Hence, the answer is 2.532 rad./s.

12 Orbital motion

12.1 Introduction

We have spent this course exploring the theory of motion first outlined by Sir Isaac Newton in his *Principia* (1687). It is, therefore, interesting to discuss the particular application of this theory which made Newton an international celebrity, and which profoundly and permanently changed humankind's outlook on the Universe. This application is, of course, the motion of the Solar System.

12.2 Historical background

Humankind has always been fascinated by the night sky, and, in particular, by the movements of the Sun, the Moon, and the objects which the ancient Greeks called *plantai* ("wanderers"), and which we call *planets*. In ancient times, much of this interest was of a practical nature. The Sun and the Moon were important for determining the calendar, and also for navigation. Moreover, the planets were vital to astrology: *i.e.*, the belief—almost universally prevalent in the ancient world—that the positions of the planets in the sky could be used to foretell important events.

Actually, there were only seven "wandering" heavenly bodies visible to ancient peoples: the Sun, the Moon, and the five planets—Mercury, Venus, Mars, Jupiter, and Saturn. The ancients believed that the stars were fixed to a "celestial sphere" which formed the outer boundary of the Universe. However, it was recognized that the wandering bodies were located *within* this sphere: *e.g.*, because the Moon clearly passes in front of, and blocks the light from, stars in its path. It was also recognized that some bodies were closer to the Earth than others. For instance, ancient astronomers noted that the Moon occasionally passes in front of the Sun and each of the planets. Moreover, Mercury and Venus can sometimes be seen to transit in front of the Sun.

The first scientific model of the Solar System was outlined by the Greek philosopher Eudoxas of Cnidus (409–356 BC). According to this model, the Sun, the Moon, and the planets all execute uniform circular orbits around the Earth—which is fixed, and non-rotating. The order of the orbits is as follows: Moon, Mercury, Venus, Sun, Mars, Jupiter, Saturn—with the Moon closest to the Earth. For obvious reasons, Eudoxas' model became known as the *geocentric* model of the Solar System. Note that orbits are circular in this model for philosophical reasons. The ancients believed the heavens to be the realm of perfection. Since a circle is the most "perfect" imaginable shape, it follows that heavenly objects must execute circular orbits.

A second Greek philosopher, Aristarchus of Samos (310–230 BC), proposed an alternative model in which the Earth and the planets execute uniform circular orbits around the Sun—which is fixed. Moreover, the Moon orbits around the Earth, and the Earth rotates daily about a North-South axis. The order of the planetary orbits is as follows: Mercury, Venus, Earth, Mars, Jupiter, Saturn—with Mercury closest to the Sun. This model became known as the *heliocentric* model of the Solar System.

The heliocentric model was generally rejected by the ancient philosophers for three main reasons:

1. If the Earth is rotating about its axis, and orbiting around the Sun, then the Earth must be in motion. However, we cannot "feel" this motion. Nor does this motion give rise to any obvious observational consequences. Hence, the Earth must be stationary.

2. If the Earth is executing a circular orbit around the Sun then the positions of the stars should be slightly different when the Earth is on opposite sides of the Sun. This effect is known as *parallax*. Since no stellar parallax is observable (at least, with the naked eye), the Earth must be stationary. In order to appreciate the force of this argument, it is important to realize that ancient astronomers did not suppose the stars to be significantly further away from the Earth than the planets. The celestial sphere was assumed to lie just beyond the orbit of Saturn.

3. The geocentric model is far more philosophically attractive than the helio-

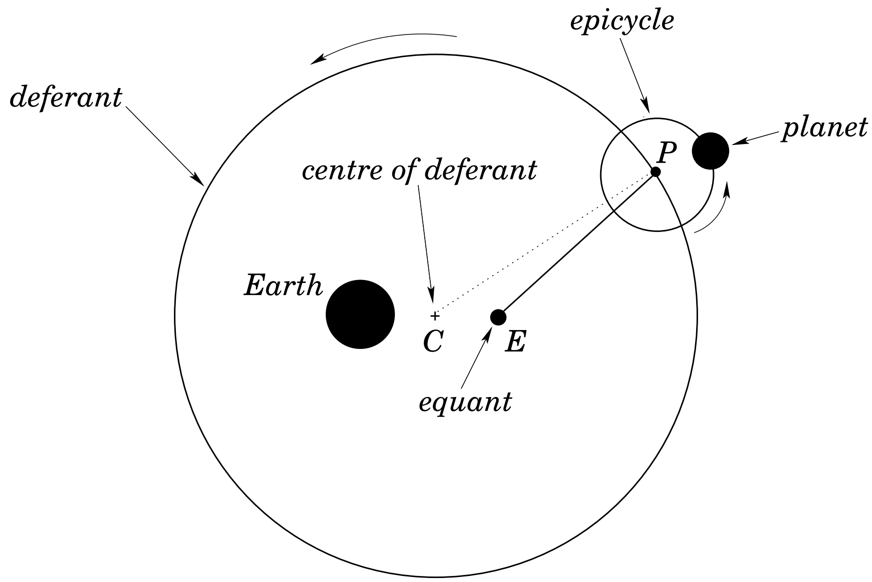


Figure 100: *The Ptolemaic system.*

centric model, since in the former model the Earth occupies a privileged position in the Universe.

The geocentric model was first converted into a proper scientific theory, capable of accurate predictions, by the Alexandrian philosopher Claudius Ptolemy (85–165 AD). The theory that Ptolemy proposed in his famous book, now known as the *Almagest*, remained the dominant scientific picture of the Solar System for over a millennium. Basically, Ptolemy acquired and extended the extensive set of planetary observations of his predecessor Hipparchus, and then constructed a geocentric model capable of accounting for them. However, in order to fit the observations, Ptolemy was forced to make some significant modifications to the original model of Eudoxas. Let us discuss these modifications.

First, we need to introduce some terminology. As shown in Fig. 100, *deferants* are large circles centred on the Earth, and *epicyles* are small circles whose centres move around the circumferences of the deferants. In the Ptolemaic system, instead of traveling around deferants, the planets move around the circumference of epicycles, which, in turn, move around the circumference of deferants. Ptolemy found, however, that this modification was insufficient to completely account for all of his data. Ptolemy's second modification to Eudoxas' model was to displace the Earth slightly from the common centre of the deferants. Moreover, Ptolemy assumed that the Sun, Moon, and planets rotate uniformly about an imaginary point, called the *equant*, which is displaced an equal distance in the opposite direction to the Earth from the centre of the deferants. In other words, Ptolemy assumed that the line EP, in Fig. 100, rotates uniformly, rather than the line CP.

Figure 101 shows more details of the Ptolemaic model.[2] Note that this diagram is not drawn to scale, and the displacement of the Earth from the centre of the deferants has been omitted for the sake of clarity. It can be seen that the Moon and the Sun do not possess epicycles. Moreover, the motions of the *inferior planets* (*i.e.*, Mercury and Venus) are closely linked to the motion of the Sun. In fact, the centres of the inferior planet epicycles move on an imaginary line connecting the Earth and the Sun. Furthermore, the radius vectors connecting the *superior planets* (*i.e.*, Mars, Jupiter, and Saturn) to the centres of their epicycles are always parallel to the geometric line connecting the Earth and the Sun. Note that, in addition to the motion indicated in the diagram, all of the heavenly bodies (including the stars) rotate clockwise (assuming that we are looking down on the Earth's North pole in Fig. 101) with a period of 1 day. Finally, there are epicycles within the epicycles shown in the diagram. In fact, some planets need as many as 28 epicycles to account for all the details of their motion. These subsidiary epicycles are not shown in the diagram, for the sake of clarity.

As is quite apparent, the Ptolemaic model of the Solar System is *extremely complicated*. However, it successfully accounted for the relatively crude naked eye observations made by the ancient Greeks. The Sun-linked epicyles of the inferior planets are needed to explain why these objects always remain close to the Sun in the sky. The epicycles of the superior planets are needed to account for their occasional bouts of *retrograde motion*: *i.e.*, motion in the opposite direction to their apparent direction of rotation around the Earth. Finally, the displacement of the Earth from the centre of the deferants, as well as the introduction of the equant as the centre of uniform rotation, is needed to explain why the planets speed up slightly when they are close to the Earth (and, hence, appear brighter in the night sky), and slow down when they are further away.

[2] R.A. Hatch, University of Florida, http://web.clas.ufl.edu/users/rhatch/

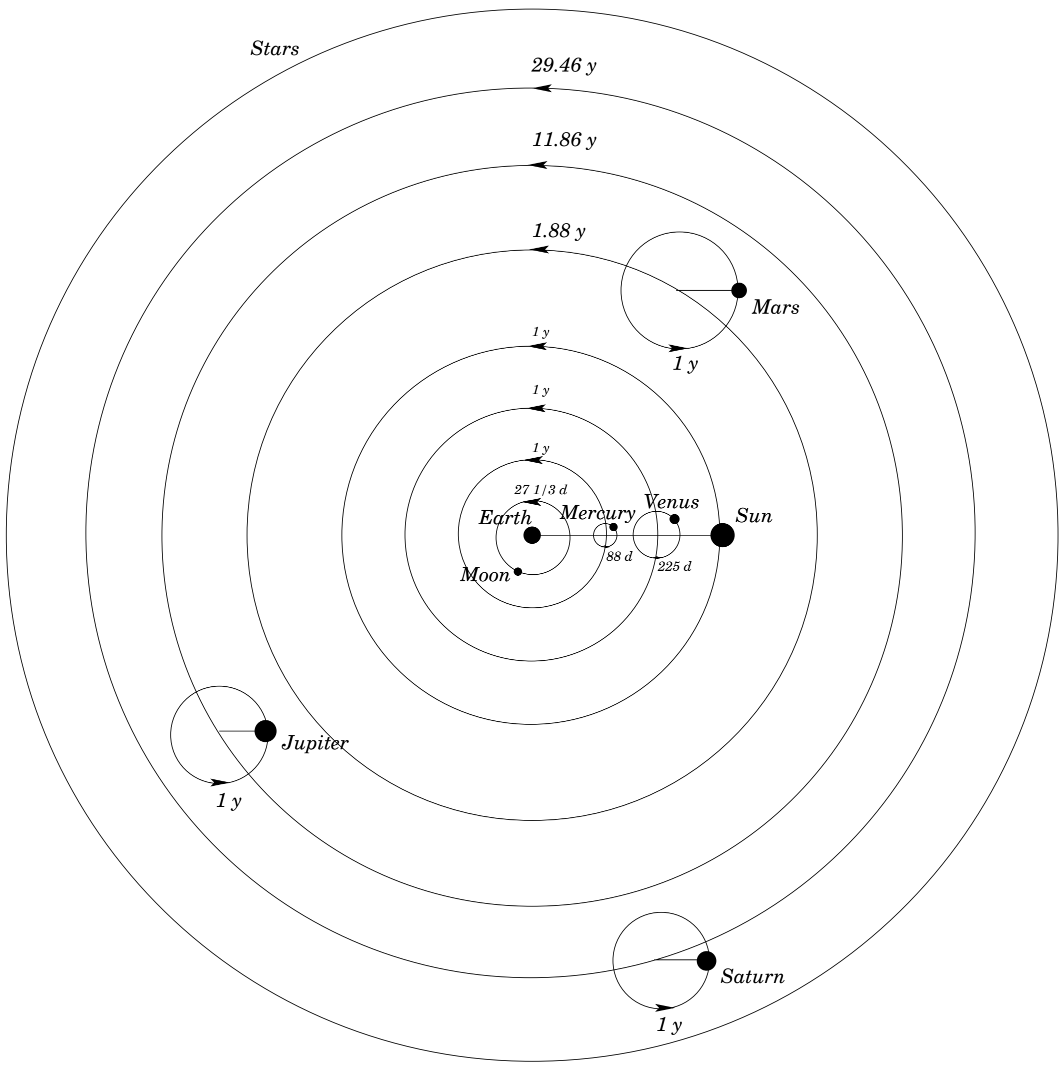


Figure 101: *The Ptolemaic model of the Solar System.*

Ptolemy's model of the Solar System was rescued from the wreck of ancient European civilization by the Roman Catholic Church, which, unfortunately, converted it into a minor article of faith, on the basis of a few references in the Bible which seemed to imply that the Earth is stationary and the Sun is moving (*e.g.*, Joshua 10:12-13, Habakkuk 3:11). Consequently, this model was not subject to proper scientific criticism for over a millennium. Having said this, few medieval or renaissance philosophers were entirely satisfied with Ptolemy's model. Their dissatisfaction focused, not on the many epicycles (which to the modern eye seem rather absurd), but on the displacement of the Earth from the centre of the deferants, and the introduction of the equant as the centre of uniform rotation. Recall, that the only reason planetary orbits are constructed from circles in Ptolemy's model is to preserve the assumed ideal symmetry of the heavens. Unfortunately, this symmetry is severely compromised when the Earth is displaced from the apparent centre of the Universe. This problem so perplexed the Polish priest-astronomer Nicolaus Copernicus (1473–1543) that he eventually decided to reject the geocentric model, and revive the heliocentric model of Aristarchus. After many years of mathematical calculations, Copernicus published a book entitled *De revolutionibus orbium coelestium* (On the revolutions of the celestial spheres) in 1543 which outlined his new heliocentric theory.

Copernicus' model is illustrated in Fig. 102. Again, this diagram is not to scale. The planets execute uniform circular orbits about the Sun, and the Moon orbits about the Earth. Finally, the Earth revolves about its axis daily. Note that there is no displacement of the Sun from the centres of the planetary orbits, and there is no equant. Moreover, in this model, the inferior planets remain close to the Sun in the sky without any special synchronization of their orbits. Furthermore, the occasional retrograde motion of the superior planets has a more natural explanation than in Ptolemy's model. Since the Earth orbits more rapidly than the superior planets, it occasionally "overtakes" them, and they appear to move backward in the night sky, in much the same manner that slow moving cars on a freeway appears to move backward to a driver overtaking them. Copernicus accounted for the lack of stellar parallax, due to the Earth's motion, by postulating that the stars were a lot further away than had previously been supposed, rendering any parallax undetectably small. Unfortunately, Copernicus insisted on retaining uni-

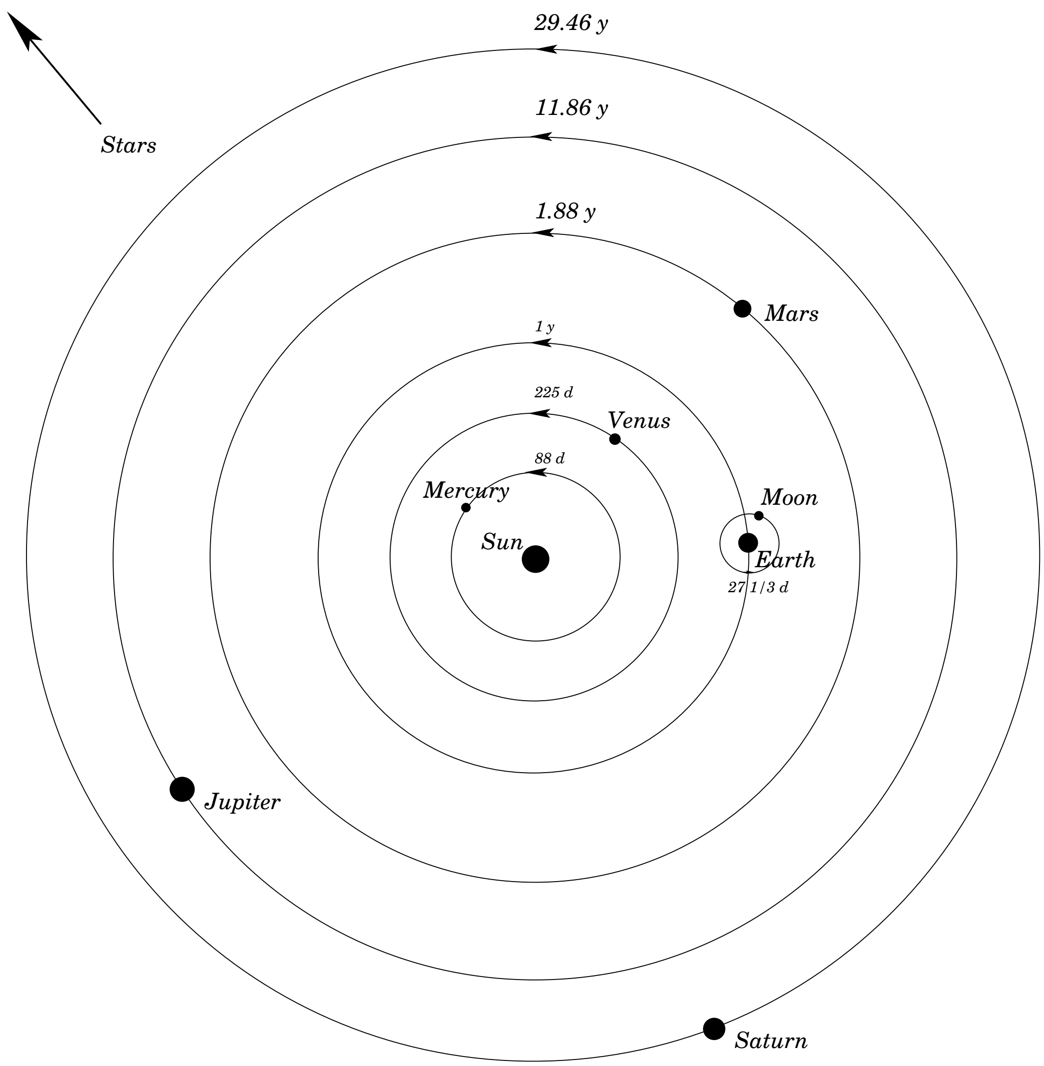


Figure 102: *The Copernican model of the Solar System.*

form circular motion in his model (after all, he was trying to construct a more symmetric model than that of Ptolemy). Consequently, Copernicus also had to resort to epicycles to fit the data. In fact, Copernicus' model ended up with more epicycles than Ptolemy's!

The real breakthrough in the understanding of planetary motion occurred—as most breakthroughs in physics occur—when better data became available. The data in question was produced by the Dane Tycho Brahe (1546–1601), who devoted his life to making naked eye astronomical observations of unprecedented accuracy and detail. This data was eventually inherited by Brahe's pupil and assistant, the German scientist Johannes Kepler (1571–1630). Kepler fully accepted Copernicus' heliocentric theory of the Solar System. Moreover, he was just as firm a believer as Copernicus in the perfection of the heavens, and the consequent need for circular motion of planetary bodies. The main difference was that Kepler's observational data was considerably better than Copernicus'. After years of fruitless effort, Kepler eventually concluded that no combination of circular deferants and epicycles could completely account for his data. At this stage, he started to think the unthinkable. Maybe, planetary motion was not circular after all? After more calculations, Kepler was eventually able to formulate three extraordinarily simple laws which completely accounted for Brahe's observations. These laws are as follows:

1. The planets move in elliptical orbits with the Sun at one focus.
2. A line from the Sun to any given planet sweeps out equal areas in equal time intervals.
3. The square of a planet's period is proportional to the cube of the planet's mean distance from the Sun.

Note that there are no epicyles or equants in Kepler's model of the Solar System.

Figure 103 illustrates Kepler's second law. Here, the ellipse represents a planetary orbit, and S represents the Sun, which is located at one of the focii of the ellipse. Suppose that the planet moves from point A to point B in the same time it takes to move from point C to point D. According the Kepler's second law,

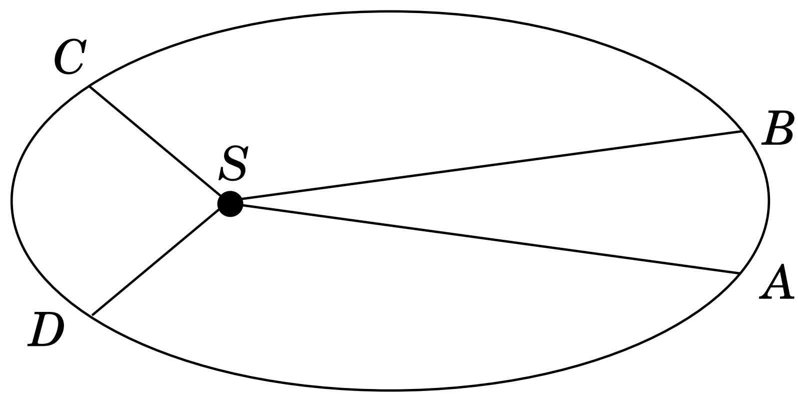


Figure 103: *Kepler's second law.*

Planet	a(AU)	T(yr)	a^3/T^2
Mercury	0.387	0.241	0.998
Venus	0.723	0.615	0.999
Earth	1.000	1.000	1.000
Mars	1.524	1.881	1.000
Jupiter	5.203	11.862	1.001
Saturn	9.516	29.458	0.993

Table 5: *Kepler's third law. Here, a is the mean distance from the Sun, measured in Astronomical Units (1 AU is the mean Earth-Sun distance), and T is the orbital period, measured in years.*

the areas of the elliptic segments ASB and CSD are equal. Note that this law basically mandates that planets speed up when they move closer to the Sun.

Table 5 illustrates Kepler's third law. The mean distance, a, and orbital period, T, as well as the ratio a^3/T^2, are listed for each of the first six planets in the Solar System. It can be seen that the ratio a^3/T^2 is indeed constant from planet to planet.

Since we have now definitely adopted a heliocentric model of the Solar System, let us discuss the ancient Greek objections to such a model, listed earlier. We have already dealt with the second objection (the absence of stellar parallax) by stating that the stars are a lot further away from the Earth than the ancient Greeks supposed. The third objection (that it is philosophically more attractive to have the Earth at the centre of the Universe) is not a valid scientific criticism. What about the first objection? If the Earth is rotating about its axis, and also orbiting the Sun, why do we not "feel" this motion? At first sight, this objection appears to have some force. After all, the rotation velocity of the Earth's surface is about 460 m/s. Moreover, the Earth's orbital velocity is approximately 30 km/s. Surely, we would notice if we were moving this rapidly? Of course, this reasoning is faulty because we know, from Newton's laws of motion, that we only "feel" the acceleration associated with motion, not the motion itself. It turns out that the acceleration at the Earth's surface due to its axial rotation is only about $0.034\,\mathrm{m/s^2}$. Moreover, the Earth's acceleration due to its orbital motion is only $0.0059\,\mathrm{m/s^2}$. Nowadays, we can detect such small accelerations, but the ancient Greeks certainly could not.

Kepler correctly formulated the three laws of planetary motion in 1619. Almost seventy years later, in 1687, Isaac Newton published his *Principia*, in which he presented, for the first time, a universal theory of motion. Newton then went on to illustrate his theory by using it to deriving Kepler's laws from first principles. Let us now discuss Newton's monumental achievement in more detail.

12.3 Gravity

There is one important question which we have avoided discussing until now. Why do objects fall towards the surface of the Earth? The ancient Greeks had a very simple answer to this question. According to Aristotle, all objects have a natural tendency to fall towards the centre of the Universe. Since the centre of the Earth coincides with the centre of the Universe, all objects also tend to fall towards the Earth's surface. So, an ancient Greek might ask, why do the planets not fall towards the Earth? Well, according to Aristotle, the planets are embedded in crystal spheres which rotate with them whilst holding them in place in the firmament. Unfortunately, Ptolemy seriously undermined this explanation by shifting the Earth slightly from the centre of the Universe. However, the *coup de grace* was delivered by Copernicus, who converted the Earth into just another planet orbiting the Sun.

So, why do objects fall towards the surface of the Earth? The first person, after Aristotle, to seriously consider this question was Sir Isaac Newton. Since

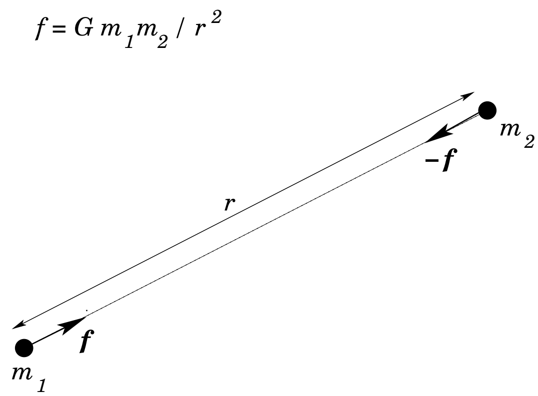


Figure 104: *Newton's law of gravity.*

the Earth is not located in a special place in the Universe, Newton reasoned, objects must be attracted toward the Earth itself. Moreover, since the Earth is just another planet, objects must be attracted towards other planets as well. In fact, all objects must exert a force of attraction on all other objects in the Universe. What intrinsic property of objects causes them to exert this attractive force—which Newton termed *gravity*—on other objects? Newton decided that the crucial property was *mass*. After much thought, he was eventually able to formulate his famous law of universal gravitation:

> Every particle in the Universe attracts every other particle with a force directly proportional to the product of their masses and inversely proportional to the square of the distance between them. The direction of the force is along the line joining the particles.

Incidentally, Newton adopted an inverse square law because he knew that this was the only type of force law which was consistent with Kepler's third law of planetary motion.

Consider two point objects of masses m_1 and m_2, separated by a distance r. As illustrated in Fig. 104, the magnitude of the force of attraction between these objects is

$$f = G\,\frac{m_1\,m_2}{r^2}. \tag{12.1}$$

The direction of the force is along the line joining the two objects.

Let $\mathbf{r}_1$ and $\mathbf{r}_2$ be the vector positions of the two objects, respectively. The vector gravitational force exerted by object 2 on object 1 can be written

$$\mathbf{f}_{12} = G\,\frac{\mathbf{r}_2 - \mathbf{r}_1}{|\mathbf{r}_2 - \mathbf{r}_1|^3}. \tag{12.2}$$

Likewise, the vector gravitational force exerted by object 1 on object 2 takes the form

$$\mathbf{f}_{21} = G\,\frac{\mathbf{r}_1 - \mathbf{r}_2}{|\mathbf{r}_1 - \mathbf{r}_2|^3} = -\mathbf{f}_{21}. \tag{12.3}$$

The constant of proportionality, G, appearing in the above formulae is called the *gravitational constant*. Newton could only estimate the value of this quantity, which was first directly measured by Henry Cavendish in 1798. The modern value of G is

$$G = 6.6726 \times 10^{-11}\,\mathrm{N\,m^2/kg^2}. \tag{12.4}$$

Note that the gravitational constant is numerically extremely small. This implies that gravity is an intrinsically weak force. In fact, gravity usually only becomes significant if at least one of the masses involved is of astronomical dimensions (*e.g.*, it is a planet, or a star).

Let us use Newton's law of gravity to account for the Earth's surface gravity. Consider an object of mass m close to the surface of the Earth, whose mass and radius are $M_\oplus = 5.97 \times 10^{24}\,\mathrm{kg}$ and $R_\oplus = 6.378 \times 10^6\,\mathrm{m}$, respectively. Newton proved, after considerable effort, that the gravitational force exerted by a spherical body (outside that body) is the same as that exerted by an equivalent point mass located at the body's centre. Hence, the gravitational force exerted by the Earth on the object in question is of magnitude

$$f = G\,\frac{m\,M_\oplus}{R_\oplus^2}, \tag{12.5}$$

and is directed towards the centre of the Earth. It follows that the equation of motion of the object can be written

$$m\,\ddot{\mathbf{r}} = -G\,\frac{m\,M_\oplus}{R_\oplus^2}\,\hat{\mathbf{z}}, \tag{12.6}$$

where $\hat{\mathbf{z}}$ is a unit vector pointing straight upwards (*i.e.*, away from the Earth's centre). Canceling the factor m on either side of the above equation, we obtain

$$\ddot{\mathbf{r}} = -g_\oplus\,\hat{\mathbf{z}}, \tag{12.7}$$

where

$$g_\oplus = \frac{G\,M_\oplus}{R_\oplus^2} = \frac{(6.673\times 10^{-11})\times(5.97\times 10^{24})}{(6.378\times 10^6)^2} = 9.79\,\mathrm{m/s^2}. \tag{12.8}$$

Thus, we conclude that all objects on the Earth's surface, irrespective of their mass, accelerate straight down (*i.e.*, towards the Earth's centre) with a constant acceleration of $9.79\,\mathrm{m/s^2}$. This estimate for the acceleration due to gravity is slightly off the conventional value of $9.81\,\mathrm{m/s^2}$ because the Earth is actually not quite spherical.

Since Newton's law of gravitation is universal, we immediately conclude that any spherical body of mass M and radius R possesses a surface gravity g given by the following formula:

$$\frac{g}{g_\oplus} = \frac{M/M_\oplus}{(R/R_\oplus)^2}. \tag{12.9}$$

Table 6 shows the surface gravity of various bodies in the Solar System, estimated using the above expression. It can be seen that the surface gravity of the Moon is only about one fifth of that of the Earth. No wonder Apollo astronauts were able to jump so far on the Moon's surface! Prospective Mars colonists should note that they will only weigh about a third of their terrestrial weight on Mars.

12.4 Gravitational potential energy

We saw earlier, in Sect. 5.5, that gravity is a conservative force, and, therefore, has an associated potential energy. Let us obtain a general formula for this energy. Consider a point object of mass m, which is a radial distance r from another point object of mass M. The gravitational force acting on the first mass is of magnitude $f = G\,M/r^2$, and is directed towards the second mass. Imagine that the first mass moves radially away from the second mass, until it reaches infinity. What

Body	$M/M_\oplus$	$R/R_\oplus$	$g/g_\oplus$
Sun	3.33×10^5	109.0	28.1
Moon	0.0123	0.273	0.17
Mercury	0.0553	0.383	0.38
Venus	0.816	0.949	0.91
Earth	1.000	1.000	1.000
Mars	0.108	0.533	0.38
Jupiter	318.3	11.21	2.5
Saturn	95.14	9.45	1.07

Table 6: *The mass, M, radius, R, and surface gravity, g, of various bodies in the Solar System. All quantities are expressed as fractions of the corresponding terrestrial quantity.*

is the change in the potential energy of the first mass associated with this shift? According to Eq. (5.33),

$$U(\infty) - U(r) = -\int_r^\infty [-f(r)]\,dr. \tag{12.10}$$

There is a minus sign in front of f because this force is oppositely directed to the motion. The above expression can be integrated to give

$$U(r) = -\frac{G\,M\,m}{r}. \tag{12.11}$$

Here, we have adopted the convenient normalization that the potential energy at infinity is zero. According to the above formula, the gravitational potential energy of a mass m located a distance r from a mass M is simply $-G\,M\,m/r$.

Consider an object of mass m moving close to the Earth's surface. The potential energy of such an object can be written

$$U = -\frac{G\,M_\oplus\,m}{R_\oplus + z}, \tag{12.12}$$

where $M_\oplus$ and $R_\oplus$ are the mass and radius of the Earth, respectively, and z is the vertical height of the object above the Earth's surface. In the limit that $z \ll R_\oplus$, the above expression can be expanded using the binomial theorem to give

$$U \simeq -\frac{G\,M_\oplus\,m}{R_\oplus} + \frac{G\,M_\oplus\,m}{R_\oplus^2}\,z, \tag{12.13}$$

Since potential energy is undetermined to an arbitrary additive constant, we could just as well write

$$U \simeq m\,g\,z, \tag{12.14}$$

where $g = G\,M_\oplus/R_\oplus^2$ is the acceleration due to gravity at the Earth's surface [see Eq. (12.8)]. Of course, the above formula is equivalent to the formula (5.3) derived earlier on in this course.

For an object of mass m and speed v, moving in the gravitational field of a fixed object of mass M, we expect the total energy,

$$E = K + U, \tag{12.15}$$

to be a constant of the motion. Here, the kinetic energy is written $K = (1/2)\,m\,v^2$, whereas the potential energy takes the form $U = -G\,M\,m/r$. Of course, r is the distance between the two objects. Suppose that the fixed object is a sphere of radius R. Suppose, further, that the second object is launched from the surface of this sphere with some velocity v_{esc} which is such that it *only just* escapes the sphere's gravitational influence. After the object has escaped, it is a long way away from the sphere, and hence $U = 0$. Moreover, if the object only just escaped, then we also expect $K = 0$, since the object will have expended all of its initial kinetic energy escaping from the sphere's gravitational well. We conclude that our object possesses zero net energy: *i.e.*, $E = K + U = 0$. Since E is a constant of the motion, it follows that at the launch point

$$E = \frac{1}{2}\,m\,v_{esc}^2 - \frac{G\,M\,m}{R} = 0. \tag{12.16}$$

This expression can be rearranged to give

$$v_{esc} = \sqrt{\frac{2\,G\,M}{R}}. \tag{12.17}$$

The quantity v_{esc} is known as the *escape velocity*. Objects launched from the surface of the sphere with velocities exceeding this value will eventually escape from the sphere's gravitational influence. Otherwise, the objects will remain in orbit around the sphere, and may eventually strike its surface. Note that the escape velocity is independent of the object's mass and launch direction (assuming that it is not straight into the sphere).

The escape velocity for the Earth is

$$v_{esc} = \sqrt{\frac{2\,G\,M_\oplus}{R_\oplus}} = \sqrt{\frac{2\times(6.673\times10^{-11})\times(5.97\times10^{24})}{6.378\times10^{6}}} = 11.2\,\text{km/s}. \quad (12.18)$$

Clearly, NASA must launch deep space probes from the surface of the Earth with velocities which exceed this value if they are to have any hope of eventually reaching their targets.

12.5 Satellite orbits

Consider an artificial satellite executing a circular orbit of radius r around the Earth. Let ω be the satellite's orbital angular velocity. The satellite experiences an acceleration towards the Earth's centre of magnitude $\omega^2\,r$. Of course, this acceleration is provided by the gravitational attraction between the satellite and the Earth, which yields an acceleration of magnitude $G\,M_\oplus/r^2$. It follows that

$$\omega^2\,r = \frac{G\,M_\oplus}{r^2}. \quad (12.19)$$

Suppose that the satellite's orbit lies in the Earth's equatorial plane. Moreover, suppose that the satellite's orbital angular velocity just matches the Earth's angular velocity of rotation. In this case, the satellite will appear to hover in the *same place* in the sky to a stationary observer on the Earth's surface. A satellite with this singular property is known as a *geostationary satellite*.

Virtually all of the satellites used to monitor the Earth's weather patterns are geostationary in nature. Communications satellites also tend to be geostationary. Of course, the satellites which beam satellite-TV to homes across the world *must* be geostationary—otherwise, you would need to install an expensive tracking antenna on top of your house in order to pick up the transmissions. Incidentally, the person who first envisaged rapid global telecommunication via a network of geostationary satellites was the science fiction writer Arthur C. Clarke in 1945.

Let us calculate the orbital radius of a geostationary satellite. The angular velocity of the Earth's rotation is

$$\omega = \frac{2\pi}{24\times 60\times 60} = 7.27\times 10^{-5}\,\text{rad./s.} \tag{12.20}$$

It follows from Eq. (12.19) that

$$\begin{aligned} r_{geo} &= \left(\frac{G\,M_\oplus}{\omega^2}\right)^{1/3} = \left(\frac{(6.673\times 10^{-11})\times(5.97\times 10^{24})}{(7.27\times 10^{-5})^2}\right)^{1/3} \\ &= 4.22\times 10^7\,\text{m} = 6.62\,R_\oplus. \end{aligned} \tag{12.21}$$

Thus, a geostationary satellite must be placed in a circular orbit whose radius is *exactly* 6.62 times the Earth's radius.

12.6 Planetary orbits

Let us now see whether we can use Newton's universal laws of motion to derive Kepler's laws of planetary motion. Consider a planet orbiting around the Sun. It is convenient to specify the planet's instantaneous position, with respect to the Sun, in terms of the *polar coordinates* r and θ. As illustrated in Fig. 105, r is the radial distance between the planet and the Sun, whereas θ is the angular bearing of the planet, from the Sun, measured with respect to some arbitrarily chosen direction.

Let us define two unit vectors, $\mathbf{e}_r$ and $\mathbf{e}_\theta$. (A unit vector is simply a vector whose length is unity.) As shown in Fig. 105, the *radial* unit vector $\mathbf{e}_r$ always points from the Sun towards the instantaneous position of the planet. Moreover, the *tangential* unit vector $\mathbf{e}_\theta$ is always normal to $\mathbf{e}_r$, in the direction of increasing θ. In Sect. 7.5, we demonstrated that when acceleration is written in terms of polar coordinates, it takes the form

$$\mathbf{a} = a_r\,\mathbf{e}_r + a_\theta\,\mathbf{e}_\theta, \tag{12.22}$$

where

$$a_r = \ddot{r} - r\,\dot{\theta}^2, \tag{12.23}$$

$$a_\theta = r\,\ddot{\theta} + 2\,\dot{r}\,\dot{\theta}. \tag{12.24}$$

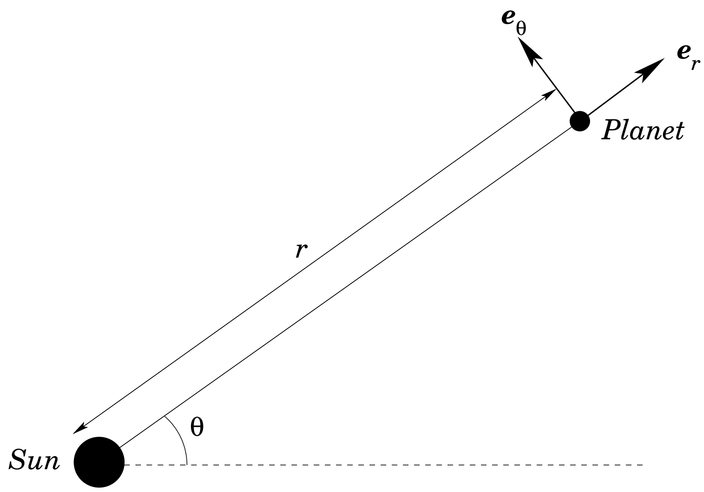


Figure 105: *A planetary orbit.*

These expressions are more complicated that the corresponding cartesian expressions because the unit vectors $\mathbf{e}_r$ and $\mathbf{e}_\theta$ *change direction* as the planet changes position.

Now, the planet is subject to a single force: *i.e.*, the force of gravitational attraction exerted by the Sun. In polar coordinates, this force takes a particularly simple form (which is why we are using polar coordinates):

$$\mathbf{f} = -\frac{G\,M_\odot\,m}{r^2}\,\mathbf{e}_r. \tag{12.25}$$

The minus sign indicates that the force is directed towards, rather than away from, the Sun.

According to Newton's second law, the planet's equation of motion is written

$$m\,\mathbf{a} = \mathbf{f}. \tag{12.26}$$

The above four equations yield

$$\ddot{r} - r\,\dot{\theta}^2 = -\frac{G\,M_\odot}{r^2}, \tag{12.27}$$

$$r\,\ddot{\theta} + 2\,\dot{r}\,\dot{\theta} = 0. \tag{12.28}$$

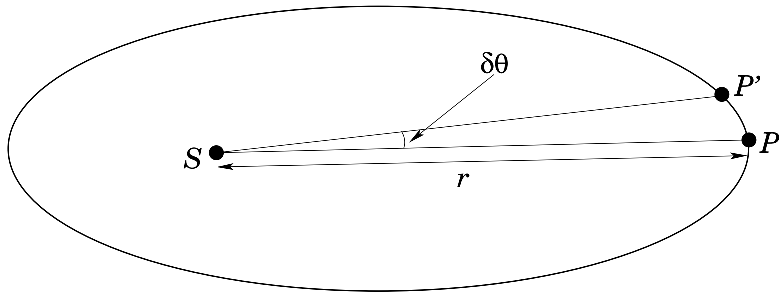


Figure 106: *The origin of Kepler's second law.*

Equation (12.28) reduces to

$$\frac{d}{dt}\,(r^2\,\dot{\theta}) = 0, \tag{12.29}$$

or

$$r^2\,\dot{\theta} = h, \tag{12.30}$$

where h is a *constant of the motion*. What is the physical interpretation of h? Recall, from Sect. 9.2, that the angular momentum vector of a point particle can be written

$$\mathbf{l} = m\,\mathbf{r}\times\mathbf{v}. \tag{12.31}$$

For the case in hand, $\mathbf{r} = r\,\mathbf{e}_r$ and $\mathbf{v} = \dot{r}\,\mathbf{e}_r + r\,\dot{\theta}\,\mathbf{e}_\theta$ [see Sect. 7.5]. Hence,

$$l = m\,r\,v_\theta = m\,r^2\,\dot{\theta}, \tag{12.32}$$

yielding

$$h = \frac{l}{m}. \tag{12.33}$$

Clearly, h represents the *angular momentum* (per unit mass) of our planet around the Sun. Angular momentum is conserved (*i.e.*, h is constant) because the force of gravitational attraction between the planet and the Sun exerts *zero torque* on the planet. (Recall, from Sect. 9, that torque is the rate of change of angular momentum.) The torque is zero because the gravitational force is *radial* in nature: *i.e.*, its line of action passes through the Sun, and so its associated lever arm is of length zero.

The quantity h has another physical interpretation. Consider Fig. 106. Suppose that our planet moves from P to P' in the short time interval δt. Here, S

represents the position of the Sun. The lines SP and SP' are both approximately of length r. Moreover, using simple trigonometry, the line PP' is of length $r\,\delta\theta$, where $\delta\theta$ is the small angle through which the line joining the Sun and the planet rotates in the time interval δt. The area of the triangle PSP' is approximately

$$\delta A = \frac{1}{2} \times r\,\delta\theta \times r: \tag{12.34}$$

i.e., half its base times its height. Of course, this area represents the area swept out by the line joining the Sun and the planet in the time interval δt. Hence, the rate at which this area is swept is given by

$$\lim_{\delta t\to 0} \frac{\delta A}{\delta t} = \frac{1}{2}\, r^2 \lim_{\delta t\to 0} \frac{\delta\theta}{\delta t} = \frac{r^2\,\dot{\theta}}{2} = \frac{h}{2}. \tag{12.35}$$

Clearly, the fact that h is a constant of the motion implies that the line joining the planet and the Sun sweeps out area at a *constant rate*: *i.e.*, the line sweeps equal areas in equal time intervals. But, this is just Kepler's second law. We conclude that Kepler's second law of planetary motion is a direct manifestation of *angular momentum conservation*.

Let

$$r = \frac{1}{u}, \tag{12.36}$$

where $u(t) \equiv u(\theta)$ is a new radial variable. Differentiating with respect to t, we obtain

$$\dot{r} = -\frac{\dot{u}}{u^2} = -\frac{\dot{\theta}}{u^2}\frac{du}{d\theta} = -h\,\frac{du}{d\theta}. \tag{12.37}$$

The last step follows from the fact that $\dot{\theta} = h\,u^2$. Differentiating a second time with respect to t, we obtain

$$\ddot{r} = -h\,\frac{d}{dt}\left(\frac{du}{d\theta}\right) = -h\,\dot{\theta}\,\frac{d^2u}{d\theta^2} = -h^2\,u^2\,\frac{d^2u}{d\theta^2}. \tag{12.38}$$

Equations (12.27) and (12.38) can be combined to give

$$\frac{d^2u}{d\theta^2} + u = \frac{G\,M_\odot}{h^2}. \tag{12.39}$$

This equation possesses the fairly obvious general solution

$$u = A\,\cos(\theta-\theta_0) + \frac{G\,M_\odot}{h^2}, \tag{12.40}$$

where A and θ_0 are arbitrary constants.

The above formula can be inverted to give the following simple orbit equation for our planet:

$$r = \frac{1}{A\,\cos(\theta-\theta_0) + G\,M_\odot/h^2}. \tag{12.41}$$

The constant θ_0 merely determines the orientation of the orbit. Since we are only interested in the orbit's *shape*, we can set this quantity to zero without loss of generality. Hence, our orbit equation reduces to

$$r = r_0\,\frac{1+e}{1+e\,\cos\theta}, \tag{12.42}$$

where

$$e = \frac{A\,h^2}{G\,M_\odot}, \tag{12.43}$$

and

$$r_0 = \frac{h^2}{G\,M_\odot\,(1+e)}. \tag{12.44}$$

Formula (12.42) is the standard equation of an *ellipse* (assuming $e < 1$), with the origin at a focus. Hence, we have now proved Kepler's first law of planetary motion. It is clear that r_0 is the radial distance at $\theta = 0$. The radial distance at $\theta = \pi$ is written

$$r_1 = r_0\,\frac{1+e}{1-e}. \tag{12.45}$$

Here, r_0 is termed the *perihelion* distance (*i.e.*, the closest distance to the Sun) and r_1 is termed the *aphelion* distance (*i.e.*, the furthest distance from the Sun). The quantity

$$e = \frac{r_1 - r_0}{r_1 + r_0} \tag{12.46}$$

is termed the *eccentricity* of the orbit, and is a measure of its departure from circularity. Thus, $e = 0$ corresponds to a purely circular orbit, whereas $e \rightarrow$

Planet	e
Mercury	0.206
Venus	0.007
Earth	0.017
Mars	0.093
Jupiter	0.048
Saturn	0.056

Table 7: *The orbital eccentricities of various planets in the Solar System.*

1 corresponds to a highly elongated orbit. As specified in Tab. 7, the orbital eccentricities of all of the planets (except Mercury) are fairly small.

According to Eq. (12.35), a line joining the Sun and an orbiting planet sweeps area at the constant rate $h/2$. Let T be the planet's orbital period. We expect the line to sweep out the *whole area* of the ellipse enclosed by the planet's orbit in the time interval T. Since the area of an ellipse is $\pi\, a\, b$, where a and b are the *semi-major* and *semi-minor* axes, we can write

$$T = \frac{\pi\, a\, b}{h/2}. \tag{12.47}$$

Incidentally, Fig. 107 illustrates the relationship between the aphelion distance, the perihelion distance, and the semi-major and semi-minor axes of a planetary orbit. It is clear, from the figure, that the semi-major axis is just the mean of the aphelion and perihelion distances: *i.e.*,

$$a = \frac{r_0 + r_1}{2}. \tag{12.48}$$

Thus, a is essentially the planet's mean distance from the Sun. Finally, the relationship between a, b, and the eccentricity, e, is given by the well-known formula

$$\frac{b}{a} = \sqrt{1 - e^2}. \tag{12.49}$$

This formula can easily be obtained from Eq. (12.42).

Equations (12.44), (12.45), and (12.48) can be combined to give

$$a = \frac{h^2}{2\, G\, M_\odot}\left(\frac{1}{1+e} + \frac{1}{1-e}\right) = \frac{h^2}{G\, M_\odot\,(1 - e^2)}. \tag{12.50}$$

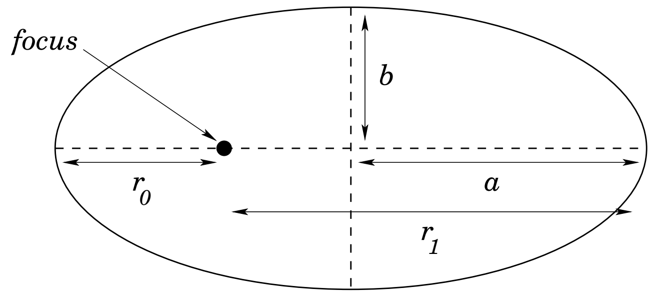


Figure 107: *Anatomy of a planetary orbit.*

It follows, from Eqs. (12.47), (12.49), and (12.50), that the orbital period can be written

$$T = \frac{2\pi}{\sqrt{G M_\odot}}\, a^{3/2}. \tag{12.51}$$

Thus, the orbital period of a planet is proportional to its mean distance from the Sun to the power 3/2—the constant of proportionality being the *same* for all planets. Of course, this is just Kepler's third law of planetary motion.

Worked example 12.1: Gravity on Callisto

Question: Callisto is the eighth of Jupiter's moons: its mass and radius are $M = 1.08 \times 10^{23}$ kg and $R = 2403$ km, respectively. What is the gravitational acceleration on the surface of this moon?

Answer: The surface gravitational acceleration on a spherical body of mass M and radius R is simply

$$g = \frac{G M}{R^2}.$$

Hence,

$$g = \frac{(6.673 \times 10^{-11}) \times (1.08 \times 10^{23})}{(2.403 \times 10^{6})^2} = 1.25\,\mathrm{m/s^2}.$$

Worked example 12.2: Acceleration of a rocket

Question: A rocket is located a distance 3.5 times the radius of the Earth above the Earth's surface. What is the rocket's free-fall acceleration?

Answer: Let $R_\oplus$ be the Earth's radius. The distance of the rocket from the centre of the Earth is $r_1 = (3.5 + 1)\, R_\oplus = 4.5\, R_\oplus$. We know that the free-fall acceleration of the rocket when its distance from the Earth's centre is $r_0 = R_\oplus$ (*i.e.*, when it is at the Earth's surface) is $g_0 = 9.81\,\mathrm{m/s^2}$. Moreover, we know that gravity is an inverse-square law (*i.e.*, $g \propto 1/r^2$). Hence, the rocket's acceleration is

$$g_1 = g_0 \left(\frac{r_0}{r_1}\right)^2 = \frac{9.81 \times 1}{(4.5)^2} = 0.484\,\mathrm{m/s^2}.$$

Worked example 12.3: Circular Earth orbit

Question: A satellite moves in a circular orbit around the Earth with speed $v = 6000\,\mathrm{m/s}$. Determine the satellite's altitude above the Earth's surface. Determine the period of the satellite's orbit. The Earth's mass and radius are $M_\oplus = 5.97 \times 10^{24}\,\mathrm{kg}$ and $R_\oplus = 6.378 \times 10^6\,\mathrm{m}$, respectively.

Answer: The acceleration of the satellite towards the centre of the Earth is v^2/r, where r is its orbital radius. This acceleration must be provided by the acceleration $G\,M_\oplus/r^2$ due to the Earth's gravitational attraction. Hence,

$$\frac{v^2}{r} = \frac{G\,M_\oplus}{r^2}.$$

The above expression can be rearranged to give

$$r = \frac{G\,M_\oplus}{v^2} = \frac{(6.673 \times 10^{-11}) \times (5.97 \times 10^{24})}{(6000)^2} = 1.107 \times 10^7\,\mathrm{m}.$$

Thus, the satellite's altitude above the Earth's surface is

$$h = r - R_\oplus = 1.107 \times 10^7 - 6.378 \times 10^6 = 4.69 \times 10^6\,\mathrm{m}.$$

The satellite's orbital period is simply

$$T = \frac{2\pi r}{v} = \frac{2 \times \pi \times (1.107 \times 10^7)}{6000} = 3.22\,\text{hours}.$$

Worked example 12.4: Halley's comet

Question: The distance of closest approach of Halley's comet to the Sun is 0.57 AU. (1 AU is the mean Earth-Sun distance.) The greatest distance of the comet from the Sun is 35 AU. The comet's speed at closest approach is 54 km/s. What is its speed when it is furthest from the Sun?

Answer: At perihelion and aphelion, the comet's velocity is perpendicular to its position vector from the Sun. Hence, at these two special points, the comet's angular momentum (around the Sun) takes the particularly simple form

$$l = m\,r\,u.$$

Here, m is the comet's mass, r is its distance from the Sun, and u is its speed. According to Kepler's second law, the comet orbits the Sun with *constant* angular momentum. Hence, we can write

$$r_0\,u_0 = r_1\,u_1,$$

where r_0 and u_0 are the perihelion distance and speed, respectively, and r_1 and u_1 are the corresponding quantities at aphelion. We are told that $r_0 = 0.57$ AU, $r_1 = 35$ AU, and $u_0 = 54$ km/s. It follows that

$$u_1 = \frac{u_0\,r_0}{r_1} = \frac{54 \times 0.57}{35} = 0.879\,\text{km/s}.$$

Worked example 12.5: Mass of star

Question: A planet is in circular orbit around a star. The period and radius of the orbit are $T = 4.3 \times 10^7$ s and $r = 2.34 \times 10^{11}$ m, respectively. Calculate the mass of the star.

Answer: Let ω be the planet's orbital angular velocity. The planet accelerates towards the star with acceleration $\omega^2 r$. The acceleration due to the star's gravitational attraction is $G M_*/r^2$, where M_* is the mass of the star. Equating these accelerations, we obtain

$$\omega^2 r = \frac{G M_*}{r^2}.$$

Now,

$$T = \frac{2\pi}{\omega}.$$

Hence, combining the previous two expressions, we get

$$M_* = \frac{4\pi^2 r^3}{G T^2}.$$

Thus, the mass of the star is

$$M_* = \frac{4 \times \pi^2 \times (2.34 \times 10^{11})^3}{(6.673 \times 10^{-11}) \times (4.3 \times 10^7)^2} = 4.01 \times 10^{30}\,\text{kg}.$$

Worked example 12.6: Launch energy

Question: What is the minimum energy required to launch a probe of mass $m = 120\,\text{kg}$ into outer space? The Earth's mass and radius are $M_\oplus = 5.97 \times 10^{24}\,\text{kg}$ and $R_\oplus = 6.378 \times 10^6\,\text{m}$, respectively.

Answer: The energy which must be given to the probe should just match the probe's gain in potential energy as it travels from the Earth's surface to outer space. By definition, the probe's potential energy in outer space is zero. The potential energy of the probe at the Earth's surface is

$$U = -\frac{G M_\oplus m}{R_\oplus} = \frac{(6.673 \times 10^{-11}) \times (5.97 \times 10^{24}) \times 120}{(6.378 \times 10^6)} = -7.495 \times 10^9\,\text{J}.$$

Thus, the gain in potential energy, which is the same as the minimum launch energy, is $7.495 \times 10^9\,\text{J}$.

13 Wave motion

13.1 Introduction

Waves are small amplitude perturbations which propagate through continuous media: *e.g.*, gases, liquids, solids, or—in the special case of electromagnetic waves—a vacuum. Wave motion is a combination of oscillatory and translational motion. Waves are important because they are the means through which virtually all information regarding the outside world is transmitted to us. For instance, we hear things via sound waves propagating through the air, and we see things via light waves. Now, the physical mechanisms which underlie sound and light wave propagation are completely different. Nevertheless, sound and light waves possesses a number of common properties which are intrinsic to wave motion itself. In this section, we shall concentrate on the *common* properties of waves, rather than those properties which are peculiar to particular wave types.

13.2 Waves on a stretched string

Probably the simplest type of wave is that which propagates down a stretched string. Consider a straight string which is stretched such that it is under uniform tension T. Let the string run along the x-axis. Suppose that the string is subject to a *small amplitude* displacement, in the y-direction, which can *vary* along its length. Let $y(x,t)$ be the string's displacement at position x and time t. What is the equation of motion for $y(x,t)$?

Consider an infinitesimal segment of the string which extends from $x - \delta x/2$ to $x + \delta x/2$. As shown in Fig. 108, this segment is subject to opposing tension forces, T, at its two ends, which act along the local tangent line to the string. Here, we are assuming that the string displacement remains sufficiently small that the tension does not vary in magnitude along the string. Suppose that the local tangent line to the string subtends angles $\delta\theta_1$ and $\delta\theta_2$ with the x-axis at $x - \delta x/2$ and $x + \delta x/2$, respectively—as shown in Fig. 108. Note that these angles are written as infinitesimal quantities because the string displacement is

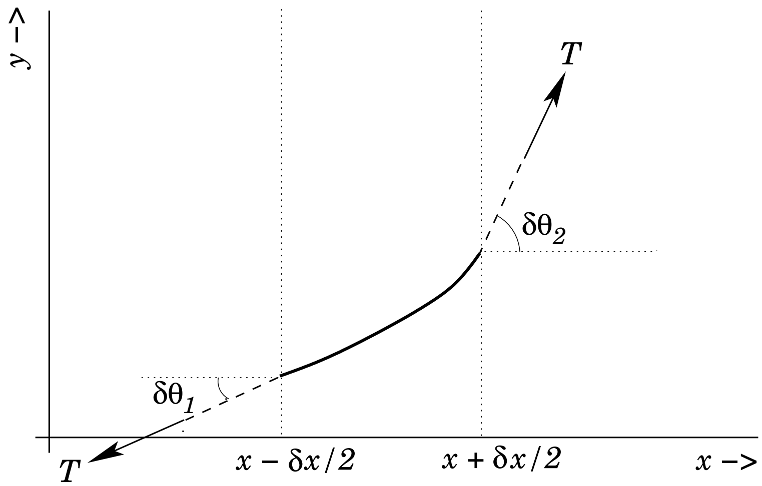


Figure 108: *Forces acting on a segment of a stretched string.*

assumed to be infinitesimally small, which implies that the string is everywhere almost parallel with the x-axis (the string displacement is greatly exaggerated in Fig. 108, for the sake of clarity).

Consider the y-component of the string segment's equation of motion. The net force acting on the segment in the y-direction takes the form

$$f_y(x, t) = T \sin \delta\theta_2 - T \sin \delta\theta_1 \simeq T (\delta\theta_2 - \delta\theta_1), \tag{13.1}$$

since $\sin\theta \simeq \theta$ when θ is small. Now, from calculus,

$$\frac{\partial y(x - \delta x/2, t)}{\partial x} = \tan \delta\theta_1 \simeq \delta\theta_1, \tag{13.2}$$

$$\frac{\partial y(x + \delta x/2, t)}{\partial x} = \tan \delta\theta_2 \simeq \delta\theta_2, \tag{13.3}$$

since the gradient, $dy(x)/dx$, of the curve $y(x)$ is equal to the tangent of the angle subtended by this curve with the x-axis. Note that $\tan\theta \simeq \theta$ when θ is small. The quantity $\partial y(x, t)/\partial x$ refers to the derivative of $y(x, t)$ with respect to x, *keeping* t *constant*—such a derivative is known as a *partial derivative*. Equations (13.1)–(13.3) can be combined to give

$$f_y(x, t) = T \left(\frac{\partial y(x + \delta x/2, t)}{\partial x} - \frac{\partial y(x - \delta x/2, t)}{\partial x} \right) = T\, \delta x\, \frac{\partial^2 y(x, t)}{\partial x^2}. \tag{13.4}$$

Here, $\partial^2 y(x,t)/\partial x^2$ is the second derivative of $y(x,t)$ with respect to x, keeping t constant.

Suppose that the string has a mass *per unit length* μ. It follows that the y equation of motion of our string segment takes the form

$$\mu\,\delta x\,\frac{\partial^2 y(x,t)}{\partial t^2} = f_y(x,t), \tag{13.5}$$

Here, $\partial^2 y(x,t)/\partial t^2$—the second derivative of $y(x,t)$ with respect to t, keeping x constant—is the y-acceleration of the string segment at position x and time t. Equations (13.4) and (13.5) yield the final expression for the string's equation of motion:

$$\frac{\partial^2 y}{\partial t^2} = \frac{T}{\mu}\,\frac{\partial^2 y}{\partial x^2}. \tag{13.6}$$

Equation (13.6) is an example of a *wave equation*. In fact, all small amplitude waves satisfy an equation of motion of this basic form. A particular solution of this type of equation has been known for centuries: *i.e.*,

$$y(x,t) = y_0\,\cos\,(k\,x - \omega\,t), \tag{13.7}$$

where y_0, k, and ω are constants. We can demonstrate that (13.7) satisfies (13.6) by direct substitution. Thus,

$$\frac{\partial y}{\partial t} = y_0\,\omega\,\sin\,(k\,x - \omega\,t), \tag{13.8}$$

$$\frac{\partial^2 y}{\partial t^2} = -y_0\,\omega^2\,\cos\,(k\,x - \omega\,t), \tag{13.9}$$

and

$$\frac{\partial y}{\partial x} = -y_0\,k\,\sin\,(k\,x - \omega\,t), \tag{13.10}$$

$$\frac{\partial^2 y}{\partial x^2} = -y_0\,k^2\,\cos\,(k\,x - \omega\,t). \tag{13.11}$$

Substituting Eqs. (13.9) and (13.11) into Eq. (13.6), we find that the latter equation is satisfied provided

$$\frac{\omega^2}{k^2} = \frac{T}{\mu}. \tag{13.12}$$

Equation (13.7) describes a pattern of motion which is *periodic in both space and time*. This periodicity follows from the well-known periodicity property of the cosine function: namely, $\cos(\theta + 2\pi) = \cos\theta$. Thus, the wave pattern is periodic in space,

$$y(x+\lambda, t) = y(x, t), \tag{13.13}$$

with periodicity length

$$\lambda = \frac{2\pi}{k}. \tag{13.14}$$

Here, λ is known as the *wavelength*, whereas k is known as the *wavenumber*. The wavelength is the distance between successive wave peaks. The wave pattern is periodic in time,

$$y(x, t+T) = y(x, t), \tag{13.15}$$

with period

$$T = \frac{2\pi}{\omega}. \tag{13.16}$$

The wave period is the oscillation period of the wave disturbance at a given point in space. The wave *frequency* (*i.e.*, the number of cycles per second the wave pattern executes at a given point in space) is written

$$f = \frac{1}{T} = \frac{\omega}{2\pi}. \tag{13.17}$$

The quantity ω is termed the *angular frequency* of the wave. Finally, at any given point in space, the displacement y oscillates between $+y_0$ and $-y_0$ (since the maximal values of $\cos\theta$ are ± 1). Hence, y_0 corresponds to the wave *amplitude*.

Equation (13.7) also describes a sinusoidal pattern which *propagates* along the x-axis *without changing shape*. We can see this by examining the motion of the wave peaks, $y = +y_0$, which correspond to

$$k\,x - \omega\,t = n\,2\pi, \tag{13.18}$$

where n is an integer. Differentiating the above expression with respect to time, we obtain

$$\frac{dx}{dt} = \frac{\omega}{k}. \tag{13.19}$$

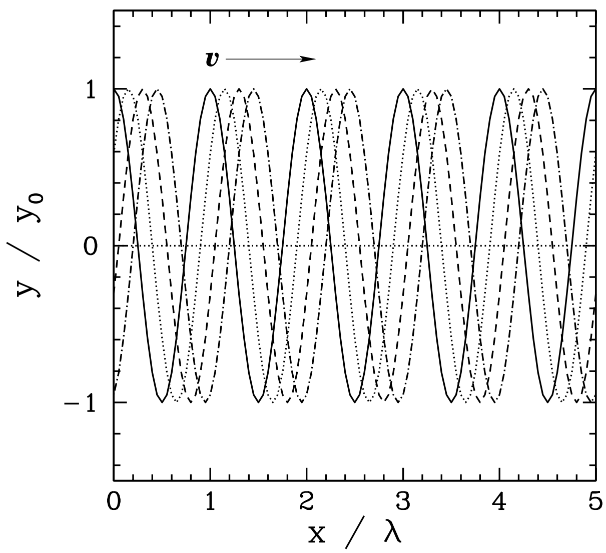


Figure 109: *A sinusoidal wave propagating down the x-axis. The solid, dotted, dashed, and dot-dashed curves show the wave displacement at four successive and equally spaced times.*

In other words, the wave peaks all propagate along the x-axis with uniform speed

$$v = \frac{\omega}{k}. \tag{13.20}$$

It is easily demonstrated that the wave troughs, $y = -y_0$, propagate with the same speed. Thus, it is fairly clear that the whole wave pattern moves with speed v—see Fig. 109. Equations (13.14), (13.17), and (13.20) yield

$$v = f\,\lambda: \tag{13.21}$$

i.e., a wave's speed is the product of its frequency and its wavelength. This is true for all types of (sinusoidal) wave.

Equations (13.12) and (13.20) imply that

$$v = \sqrt{\frac{T}{\mu}}. \tag{13.22}$$

In other words, all waves that propagate down a stretched string do so with the *same speed*. This common speed is determined by the properties of the string: *i.e.*, its tension and mass per unit length. Note, from Eq. (13.7), that the wavelength

λ is arbitrary. However, once the wavelength is specified, the wave frequency f is fixed via Eqs. (13.21) and (13.22). It follows that short wavelength waves possess high frequencies, and *vice versa*.

13.3 General waves

By analogy with the previous discussion, a general wave disturbance propagating along the x-axis satisfies

$$\frac{\partial^2 y}{\partial t^2} = v^2 \frac{\partial^2 y}{\partial x^2}, \tag{13.23}$$

where v is the common wave speed. In general, v is determined by the properties of the medium through which the wave propagates. Thus, for waves propagating along a string, the wave speed is determined by the string tension and mass per unit length; for sound waves propagating through a gas, the wave speed is determined by the gas pressure and density; and for electromagnetic waves propagating through a vacuum, the wave speed is a constant of nature: *i.e.*, $c = 3 \times 10^8\,\text{m/s}^2$.

One solution of Eq. (13.23) is

$$y(x,t) = y_0 \cos[k\,(x - v\,t)]. \tag{13.24}$$

This is interpreted as a (sinusoidal) wave of amplitude y_0 and wavelength $\lambda = 2\pi/k$ which propagates in the $+x$ direction with speed v. It is easily demonstrated that another equally good solution of Eq. (13.23) is

$$y(x,t) = y_0 \cos[k\,(x + v\,t)]. \tag{13.25}$$

This is interpreted as a (sinusoidal) wave of amplitude y_0 and wavelength $\lambda = 2\pi/k$ which propagates in the $-x$ direction with speed v.

Equation (13.23) is a *linear* partial differential equation (PDE): *i.e.*, it is invariant under the transformation $y \to a\,y + b$, where a and b are arbitrary constants. One important mathematical property of linear PDEs is that their solutions are *superposable*: *i.e.*, they can be added together and still remain solutions. Thus, if $y_1(x,t)$ and $y_2(x,t)$ are two distinct solutions of Eq. (13.23) then

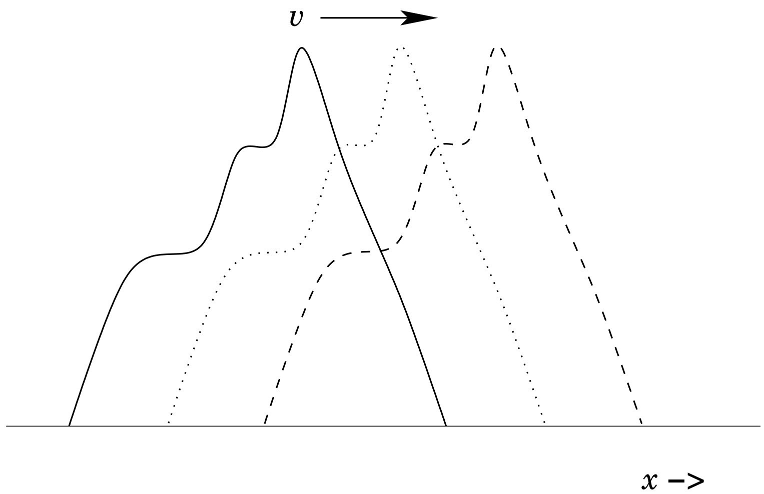


Figure 110: *A wave-pulse propagating down the x-axis. The solid, dotted, and dashed curves show the wave displacement at three successive and equally spaced times.*

$a\,y_1(x,t) + b\,y_2(x,t)$ (where a and b are arbitrary constants) is also a solution—this can be seen from inspection of Eq. (13.23). To be more exact, if

$$y_1(x,t) = a_1\,\cos\left[k_1\,(x - v\,t)\right] \tag{13.26}$$

represents a wave of amplitude a_1 and wavenumber k_1 which propagates in the $+x$ direction, and

$$y_2(x,t) = a_2\,\cos\left[k_2\,(x + v\,t)\right] \tag{13.27}$$

represents a wave of amplitude a_2 and wavenumber k_2 which propagates in the $-x$ direction, then

$$y(x,t) = y_1(x,t) + y_2(x,t) \tag{13.28}$$

is a valid solution of the wave equation, and represents the two aforementioned waves propagating in the same region *without affecting one another*.

13.4 Wave-pulses

As is easily demonstrated, the most general solution of the wave equation (13.23) is written

$$F(x - v\,t), \tag{13.29}$$

where $F(p)$ is an *arbitrary* function. The above solution is interpreted as a pulse of *arbitrary shape* which propagates in the $+x$ direction with speed v, *without changing shape*—see Fig. 110. Likewise,

$$G(x + v\,t) \tag{13.30}$$

represents another arbitrary pulse which propagates in the $-x$ direction with speed v, without changing shape. Note that, unlike our previous sinusoidal wave solutions, a general wave-pulse possesses a definite propagation speed but *does not* possess a definite wavelength or frequency.

What is the relationship between these new wave-pulse solutions and our previous sinusoidal wave solutions? It turns out that any wave-pulse can be built up from a suitable *linear superposition* of sinusoidal waves. For instance, if $F(x - v\,t)$ represents a wave-pulse propagating down the x-axis, then we can write

$$F(x - v\,t) = \int_0^\infty \bar{F}(k)\,\cos\left[k\,(x - v\,t)\right] dk, \tag{13.31}$$

where we have assumed that $F(-p) = F(p)$, for the sake of simplicity. The above formula is basically a recipe for generating the propagating wave-pulse $F(x - v\,t)$ from a suitable admixture of sinusoidal waves of definite wavelength and frequency: $\bar{F}(k)$ specifies the required amplitude of the wavelength $\lambda = 2\,\pi/k$ component. How do we determine $\bar{F}(k)$ for a given wave-pulse? Well, a mathematical result known as *Fourier's theorem* yields

$$\bar{F}(k) = \frac{2}{\pi}\int_0^\infty F(p)\,\cos\left(k\,p\right) dp, \tag{13.32}$$

The above expression essentially tells us the strength of the wavenumber k component of the wave-pulse $F(x - v\,t)$. Note that the function $\bar{F}(k)$ is known as the *Fourier spectrum* of the wave-pulse $F(x - v\,t)$.

Figures 111 and 112 show two different wave-pulses and their associated Fourier spectra. Note how, by combining sinusoidal waves of varying wavenumber in different proportions, it is possible to build up wave-pulses of completely different shape.

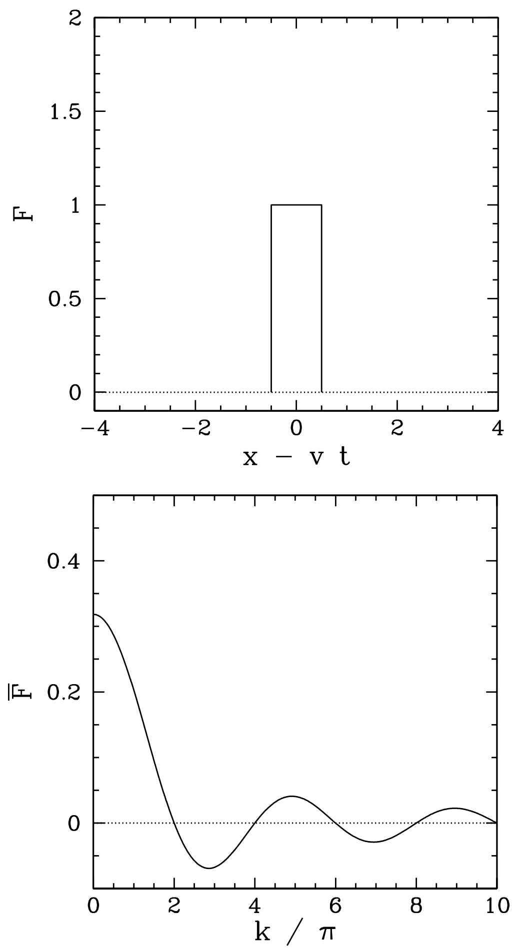


Figure 111: *A propagating wave-pulse,* $F(x - v\,t)$*, and its associated Fourier spectrum,* $\bar{F}(k)$.

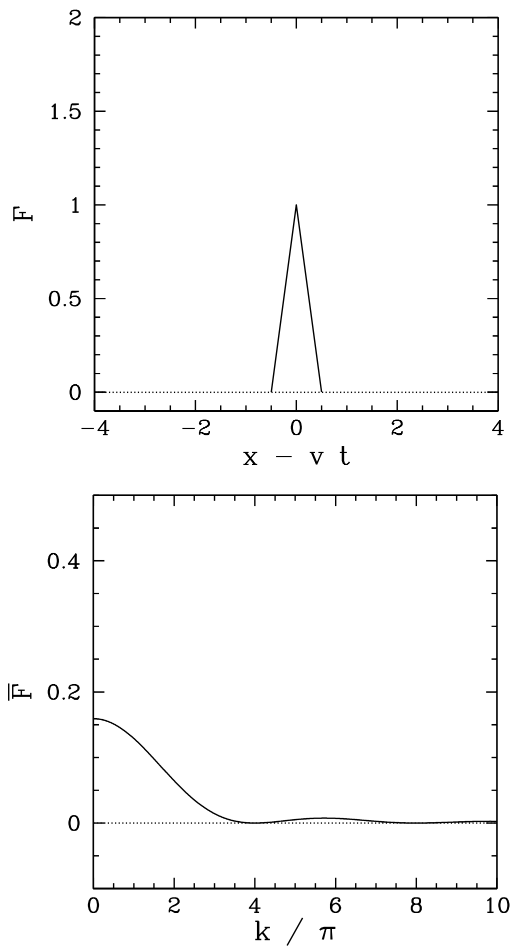


Figure 112: *A propagating wave-pulse,* $F(x - v\,t)$*, and its associated Fourier spectrum,* $\bar{F}(k)$.

13.5 Standing waves

Up to now, all of the wave solutions that we have investigated have been propagating solutions. Is it possible to construct a wave solution which does not propagate? Suppose we combine a sinusoidal wave of amplitude y_0 and wavenumber k which propagates in the $+x$ direction,

$$y_1(x,t) = y_0 \cos(k\,x - \omega\,t), \tag{13.33}$$

with a second sinusoidal wave of amplitude y_0 and wavenumber k which propagates in the $-x$ direction,

$$y_2(x,t) = y_0 \cos(k\,x + \omega\,t). \tag{13.34}$$

The net result is

$$y(x,t) = y_1(x,t) + y_2(x,t) = y_0\,[\cos(k\,x - \omega\,t) + \cos(k\,x + \omega\,t)]\,. \tag{13.35}$$

Making use of the standard trigonometric identity

$$\cos x + \cos y = 2\,\cos\left(\frac{x+y}{2}\right)\,\cos\left(\frac{x-y}{2}\right), \tag{13.36}$$

we obtain

$$y(x,t) = 2\,y_0\,\cos(k\,x)\,\cos(\omega\,t). \tag{13.37}$$

The pattern of motion specified by the above expression is illustrated in Fig. 113. It can be seen that the wave pattern *does not* propagate along the x-axis. Note, however, that the amplitude of the wave now varies with position. At certain points, called *nodes*, the amplitude is zero. At other points, called *anti-nodes*, the amplitude is maximal. The nodes are halfway between successive anti-nodes, and both nodes and anti-nodes are evenly spaced half a wavelength apart.

The standing wave shown in Fig. 113 can be thought of as the *interference pattern* generated by combining the two traveling wave solutions $y_1(x,t)$ and $y_2(x,t)$. At the anti-nodes, the waves reinforce one another, so that the oscillation amplitude becomes double that associated with each wave individually—this is termed *constructive interference*. At the nodes, the waves completely cancel one another out—this is termed *destructive interference*.

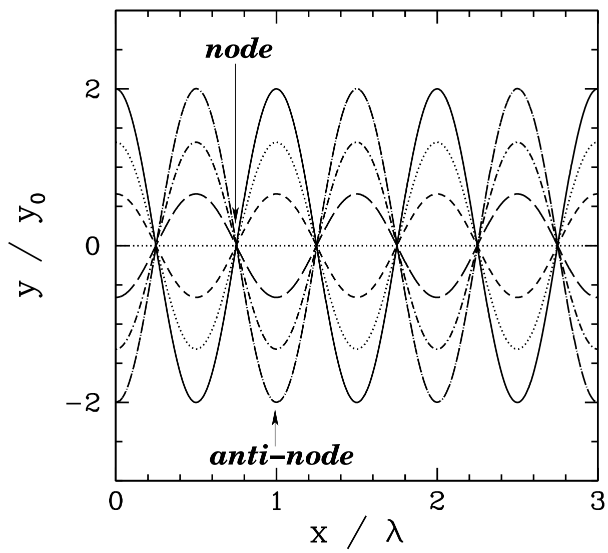


Figure 113: *A standing wave. The various curves show the wave displacement at different times.*

Most musical instruments work by exciting standing waves. For instance, stringed instruments excite standing waves on strings, whereas wind instruments excite standing waves in columns of air. Consider a guitar string of length L. Suppose that the string runs along the x-axis, and extends from $x = 0$ to $x = L$. Since the ends of the string are fixed, any wave excited on the string must satisfy the constraints

$$y(0,t) = y(L,t) = 0. \tag{13.38}$$

It is fairly clear that no propagating wave solution of the form $y_0 \cos[k\,(x \pm v\,t)]$ can satisfy these constraints. However, a standing wave can easily satisfy the constraints, provided two of its nodes coincide with the ends of the string. Since the nodes in a standing wave pattern are spaced half a wavelength apart, it follows that the wave frequency must be adjusted such that an integer number of half-wavelengths fit on the string. In other words,

$$L = n\,\frac{\lambda}{2}, \tag{13.39}$$

where $n = 1, 2, 3, \ldots$. Now, from Eqs. (13.21) and (13.22),

$$f\,\lambda = \sqrt{\frac{T}{\mu}}, \tag{13.40}$$

where T and μ are the tension and mass per unit length of the string, respectively. The above two equations can be combined to give

$$f = \frac{n}{2L}\sqrt{\frac{T}{\mu}}. \tag{13.41}$$

Thus, the standing waves that can be excited on a guitar string have frequencies f_0, $2\,f_0$, $3\,f_0$, *etc.*, which are integer multiples of

$$f_0 = \frac{1}{2L}\sqrt{\frac{T}{\mu}}. \tag{13.42}$$

These frequencies are transmitted to our ear, via sound waves which oscillate in sympathy with the guitar string, and are interpreted as musical notes. To be more exact, the frequencies correspond to notes spaced an octave apart. The frequency f_0 is termed the *fundamental* frequency, whereas the frequencies $2\,f_0$, $3\,f_0$, *etc.* are termed the *overtone harmonic* frequencies. When a guitar string is plucked an admixture of standing waves, consisting predominantly of the fundamental harmonic wave, is excited on the string. The fundamental harmonic determines the musical note which the guitar string plays. However, it is the overtone harmonics which give the note its peculiar timbre. Thus, a trumpet sounds different to a guitar, even when they are both playing the same note, because a trumpet excites a different mix of overtone harmonics than a guitar.

13.6 The Doppler effect

Consider a sinusoidal wave of wavenumber k and angular frequency ω propagating in the $+x$ direction:

$$y(x,t) = y_0\cos(k\,x - \omega\,t). \tag{13.43}$$

The wavelength and frequency of the wave, as seen by a stationary observer, are $\lambda = 2\,\pi/k$ and $f = \omega/2\,\pi$, respectively. Consider a second observer moving with uniform speed v_o in the $+x$ direction. What are the wavelength and frequency of the wave, as seen by the second observer? Well, the x-coordinate in the moving observer's frame of reference is $x' = x - v_0\,t$ (see Sect. 4.9). Of course, both observers measure the same time. Hence, in the second observer's frame of reference the wave takes the form

$$y(x', t) = y_0 \cos(k\, x' - \omega'\, t), \tag{13.44}$$

where

$$\omega' = \omega - k\, v_o. \tag{13.45}$$

Here, we have simply replaced x by $x' + v_o\, t$ in Eq. (13.43). Clearly, the moving observer sees a wave possessing the *same wavelength* (*i.e.*, the same k) but a *different frequency* (*i.e.*, a different ω) to that seen by the stationary observer. This phenomenon is called the *Doppler effect*. Since $v = \omega/k$, it follows that the wave speed is also shifted in the moving observer's frame of reference. In fact,

$$v' = v - v_o, \tag{13.46}$$

where v' is the wave speed seen by the moving observer. Finally, since $v = f\,\lambda$, and the wavelength is the same in both the moving and stationary observers' frames of reference, the wave frequency experienced by the moving observer is

$$f' = \left(1 - \frac{v_o}{v}\right) f. \tag{13.47}$$

Thus, the moving observer sees a lower frequency wave than the stationary observer. This occurs because the moving observer is traveling in the same direction as the wave, and is therefore effectively trying to catch it up. It is easily demonstrated that an observer moving in the opposite direction to a wave sees a higher frequency than a stationary observer. Hence, the general Doppler shift formula (for a moving observer and a stationary wave source) is

$$f' = \left(1 \mp \frac{v_o}{v}\right) f, \tag{13.48}$$

where the upper/lower signs correspond to the observer moving in the same/opposite direction to the wave.

Consider a stationary observer measuring a wave emitted by a source which is moving towards the observer with speed v_s. Let v be the propagation speed of the wave. Consider two neighbouring wave crests emitted by the source. Suppose that the first is emitted at time $t = 0$, and the second at time $t = T$, where $T = 1/f$ is the wave period in the frame of reference of the source. At time t, the first wave crest has traveled a distance $d_1 = v\,t$ towards the observer, whereas the second wave crest has traveled a distance $d_2 = v\,(t - T) + v_s\,T$ (measured from the position of the source at $t = 0$). Here, we have taken into account the fact that the source is a distance $v_s\,T$ closer to the observer when the second wave crest is emitted. The effective wavelength, λ', seen by the observer is the distance between neighbouring wave crests. Hence,

$$\lambda' = d_1 - d_2 = (v - v_s)\,T. \tag{13.49}$$

Since $v = f'\,\lambda'$, the effective frequency f' seen by the observer is

$$f' = \frac{f}{1 - v_s/v}, \tag{13.50}$$

where f is the wave frequency in the frame of reference of the source. We conclude that if the source is moving *towards* the observer then the wave frequency is shifted *upwards*. Likewise, if the source is moving *away* from the observer then the frequency is shifted *downwards*. This manifestation of the Doppler effect should be familiar to everyone. When an ambulance passes us on the street, its siren has a higher pitch (*i.e.*, a high frequency) when it is coming towards us than when it is moving away from us. Of course, the oscillation frequency of the siren never changes. It is the Doppler shift induced by the motion of the siren with respect to a stationary listener which causes the frequency change.

The general formula for the shift in a wave's frequency induced by relative motion of the observer and the source is

$$f' = \left(\frac{1 \mp v_o/v}{1 \pm v_s/v}\right) f, \tag{13.51}$$

where v_o is the speed of the observer, and v_s is the speed of the source. The upper/lower signs correspond to relative motion by which the observer and the source move apart/together.

Probably the most notorious use of the Doppler effect in everyday life is in police speed traps. In a speed trap, a policeman fires radar waves (*i.e.*, electromagnetic waves of centimeter wavelength) of fixed frequency at an oncoming car. These waves reflect off the car, which effectively becomes a moving source. Hence, by measuring the frequency increase of the reflected waves, the policeman can determine the car's speed.

Worked example 13.1: Piano range

Question: A piano emits sound waves whose frequencies range from $f_l = 28\,\text{Hz}$ to $f_h = 4200\,\text{Hz}$. What range of wavelengths is spanned by these waves? The speed of sound in air is $v = 343\,\text{m/s}$.

Answer: The relationship between a wave's frequency, f, wavelength, λ, and speed, v, is

$$v = f\,\lambda.$$

Hence, $\lambda = v/f$. The shortest wavelength (which corresponds to the highest frequency) is

$$\lambda_l = \frac{v}{f_h} = \frac{343}{4200} = 8.1667 \times 10^{-2}\,\text{m}.$$

The longest wavelength (which corresponds to the lowest frequency) is

$$\lambda_h = \frac{v}{f_l} = \frac{343}{28} = 12.250\,\text{m}.$$

Worked example 13.2: Middle C

Question: A steel wire in a piano has a length of $L = 0.9\,\text{m}$ and a mass of $m = 5.4\,\text{g}$. To what tension T must this wire be stretched so that its fundamental vibration corresponds to middle C: *i.e.*, the vibration possess a frequency $f = 261.6\,\text{Hz}$.

Answer: The fundamental standing wave on a stretched wire is such that the length L of the wire corresponds to half the wavelength λ of the wave. Hence,

$$\lambda = 2\,L = 1.80\,\text{m}.$$

The propagation speed of waves on the wire is given by

$$v = f\,\lambda = 261.6 \times 1.80 = 470.88\,\mathrm{m/s}.$$

Furthermore, the string's mass per unit length is

$$\mu = \frac{m}{L} = \frac{5.4 \times 10^{-3}}{0.9} = 6.00 \times 10^{-3}\,\mathrm{kg/m}.$$

Now, the relationship between the wave propagation speed, v, the mass per unit length, μ, and the tension, T, of a stretched wire is

$$v = \sqrt{\frac{T}{\mu}}.$$

Thus,

$$T = v^2\,\mu = (470.88)^2 \times 6.00 \times 10^{-3} = 1.330 \times 10^3\,\mathrm{N}.$$

Worked example 13.3: Sinusoidal wave

Question: A wave is described by

$$y = A\,\sin\,(k\,x - \omega\,t),$$

where $A = 4\,\mathrm{cm}$, $k = 2.65\,\mathrm{rad./m}$, and $\omega = 4.78\,\mathrm{rad./s}$. Moreover, x is in meters and t is in seconds. What are the wavelength, frequency, and propagation speed of the wave?

Answer: We identify A as the wave amplitude, k as the wavenumber, and ω as the angular frequency. Now, $k = 2\,\pi/\lambda$, where λ is the wavelength. Hence,

$$\lambda = \frac{2\,\pi}{k} = \frac{2 \times \pi}{2.65} = 2.371\,\mathrm{m}.$$

Furthermore, $\omega = 2\,\pi\,f$, where f is the frequency. Hence,

$$f = \frac{\omega}{2\,\pi} = \frac{4.78}{2 \times \pi} = 0.7608\,\mathrm{Hz}.$$

Finally, $v = f\,\lambda$, where v is the propagation speed. Thus,

$$v = 0.7608 \times 2.371 = 1.804\,\mathrm{m/s}.$$

Worked example 13.4: Truck passing stationary siren

Question: A truck, moving at $v_o = 80\,\mathrm{km/hr}$, passes a stationary police car whose siren has a frequency of $f = 500\,\mathrm{Hz}$. What is the frequency change heard by the truck driver as the truck passes the police car? The speed of sound is $v = 343\,\mathrm{m/s}$.

Answer: The truck's speed is

$$v_o = \frac{80 \times 1000}{3600} = 22.22\,\mathrm{m/s}.$$

When the truck is moving towards the police car, the siren's apparent frequency is

$$f_1 = \left(1 + \frac{v_o}{v}\right) f = \left(1 + \frac{22.22}{343}\right) \times 500 = 532.39\,\mathrm{Hz}.$$

When the truck is moving away from the police car, the siren's apparent frequency is

$$f_2 = \left(1 - \frac{v_o}{v}\right) f = \left(1 - \frac{22.22}{343}\right) \times 500 = 467.61\,\mathrm{Hz}.$$

Hence, the frequency shift is

$$\Delta f = f_1 - f_2 = 532.39 - 467.61 = 64.79\,\mathrm{Hz}.$$

Worked example 13.5: Ambulance and car

Question: An ambulance is traveling down a straight road at speed $v_s = 42\,\mathrm{m/s}$. The ambulance approaches a car which is traveling on the same road, in the same direction, at speed $v_o = 33\,\mathrm{m/s}$. The ambulance driver hears his/her siren at a frequency of $f = 500\,\mathrm{Hz}$. At what frequency does the driver of the car hear the siren? The speed of sound is $v = 343\,\mathrm{m/s}$.

Answer: The apparent frequency f' of a sound wave is given by

$$f' = \left(\frac{1 - v_o/v}{1 - v_s/v}\right) f,$$

where v_o is the speed of the observer (*i.e.*, the car driver), v_s is the speed of the source (*i.e.*, the ambulance), v is the speed of sound, and f is the wave frequency in the frame of reference of the source. We have chosen a minus sign in the numerator of the above formula because the observer is moving *away from* the source, leading to a *downward* Doppler shift. We have chosen a minus sign in the denominator of the above formula because the source is moving *towards* the observer, leading to a *upward* Doppler shift. Hence,

$$f' = \left(\frac{1 - 33/343}{1 - 42/343}\right) \times 500 = 514.95\,\text{Hz}.$$