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An approximate solution of two-dimensional convex piston problem

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Abstract. A new theory of shock dynamics (NTSD) has been used to study the propagation of curved shocks originating from the motion of two-dimensional convex piston of various shapes. The results have been compared with those obtained by Whitham's classical shock dynamics and by TVD version of MacCormack's finite difference scheme.

Keywords. Shock dynamics, non-linear shock ray theory, curved shock preparation.

1. Introduction

The idea of deriving an infinite system of compatibility conditions along shock rays has been used recently (Grinfel'd [1], Maslov [2], Prasad [3]). By truncating the infinite system of compatibility conditions, a new theory of shock dynamics (NTSD) has been proposed (Ravindran and Prasad [4]), which enables one to compute the amplitude and position of the shock and also to determine the flow behind the shock upto a short distance. The NTSD has been used to solve the one-dimensional accelerating or decelerating piston problem (Lazarev, Prasad and Singh [5]). It was shown that the results agree well with those obtained by Harten's TVD finite difference scheme (FDM) and it consumes less than 1 % of the computational time taken by the FDM. It has also been shown that the NTSD with at least two compatibility conditions gives a good approximate result in the case which is equivalent to the condition that the derivative of pressure in a direction normal to the shock front is positive (Ravindran, Sundar and Prasad [12]).

The derivation of compatibility conditions on a curved shock for gas dynamic equations is quite challenging. Srinivasan and Prasad [6] used Maslov's method to derive the first compatibility condition for the gas dynamic equations and Ravindran and Prasad [7] later worked out the correct form of second compatibility condition. Lazarev, Ravindran and Prasad [8] later derived the explicit forms of the first three compatibility conditions for gas dynamic equations using Grinfel'ds method.

The chief merit of the NTSD is that it takes into account the effects of the flow behind the shock to a reasonable extent. This is quite significant for the cases when the flow behind the shock is non-uniform, and the classical shock dynamics of Whitham [13, 14, 15] (which ignores these effects) becomes inapplicable. Similar theories have also been given by Best [16, 17] (whose zeroth order truncation leads to the Whitham's theory); and Anile and Russo [18, 19], who derived compatibility conditions for a general quasi- linear system of hyperbolic partial differential equations and though their theory is very important from the theoretical point of view, it is extremely cumbersome to derive even the second compatibility condition using it.

In this paper, the NTSD with the first two compatibility conditions has been used for the determination of the shape, position and strength of a shock originating from the motion of a convex body. The purpose here is to test the validity of NTSD for the case when the initial shock shape is curved as well as the distribution of the strength along the shock is non- uniform.

The geometrical and kinematical compatibility conditions, ray coordinate system and the first and second dynamic compatibility conditions for the gas dynamic equations have been described in details in [3], only a summary of the essential results is reproduced here. Also, a minor algebraic error in the second compatibility condition appearing in ref 3 has been corrected. The results obtained by NTSD have been compared with the numerical solution of the full gas-dynamic equations by flux limited TVD version of MacCormack's finite difference scheme (FDM), and also with the classical shock dynamics due to Whitham.

2. Dynamic compatibility conditions for two-dimensional problems

We consider the propagation of a shock front Ω_t in a polytropic gas with γ as the constant ratio of specific heats. Let $\mathbf{n} = (n_1, n_2)$ be the unit vector normal to Ω_t. We assume that the velocity components u, v,; the pressure p and the density ρ are $C^\infty(\mathbf{R}^3)$ except for a discontinuity of the first kind on Ω_t . We assume further that the shock front propagates in a gas in a uniform state and at rest. We denote the quantities ahead of the shock by the suffix '+' and those behind the shock by suffix '-', and define the jump in a quantity G as: $[G] = G_+ - G_-$.

The equations for two-dimensional motion of a gas are written in the form:

$$\rho_t + (u, v)\begin{pmatrix}\rho_x\\ \rho_y\end{pmatrix} + \rho(u_x + v_y) = 0 \tag{2.1}$$

$$\begin{pmatrix}u_t\\ v_t\end{pmatrix} + \begin{pmatrix}u_x & u_y\\ v_x & v_y\end{pmatrix}\begin{pmatrix}u\\ v\end{pmatrix} + \frac{1}{\rho}\begin{pmatrix}p_x\\ p_y\end{pmatrix} = 0 \tag{2.2}$$

$$p_t + (u, v)\begin{pmatrix}p_x\\ p_y\end{pmatrix} + \gamma p(u_x + v_y) = 0 \tag{2.3}$$

Taking jump of the equations of motion (2. 1)- (2. 3) across Ω_t, we get:

$$\begin{pmatrix} D_0 \\ H_0 \\ S_0 \end{pmatrix}_t + \mathbf{P} \begin{pmatrix} D_1 \\ H_1 \\ S_1 \end{pmatrix} = \kappa \begin{pmatrix} \rho_+ - D_0 \\ 0 \\ \gamma(p_+ - S_0) \end{pmatrix} \tag{2.4}$$

where

$$D_0 = [\rho], H_0 = [n_1 u + n_2 v], S_0 = [p],$$

$$D_1 = (n_1, n_2)([\rho_x], [\rho_y])^T, S_1 = (n_1, n_2)([p_x], [p_y])^T, H_1 = (n_1, n_2)(u_1, v_1)^T$$

with $u_1 = (n_1, n_2)([u_x], [u_y])^T$ and $v_1 = (n_1, n_2)([v_x], [v_y])^T$; the superscript T denotes the transpose of a column vector and the subscripts x and y denote the derivatives with respect to the corresponding coordinate variables; κ is the curvature of the surface Ω_t ; C is the shock velocity and

$$\mathbf{P} = \begin{pmatrix} -(C + H_0) & (\rho_+ - D_0) & 0 \\ 0 & -(C + H_0) & (\rho_+ - D_0)^{-1} \\ 0 & \gamma(p_+ - S_0) & -(C + H_0) \end{pmatrix} \tag{2.5}$$

To obtain the second compatibility conditions, we differentiate each of the equations (2. 1)-(2. 3) with respect to x and y respectively, take jump across Ω_t and finally take the scalar product with the unit normal vector $\mathbf{n} = (n_1, n_2)$. After using this procedure on the two components of the momentum equation, we once again take the inner product with the unit normal vector to obtain the normal component of the second compatibility condition from the momentum equation. The results can be written compactly in the vector form as:

$$\begin{pmatrix} D_1 \\ H_1 \\ S_1 \end{pmatrix}_t + \mathbf{P} \begin{pmatrix} D_2 \\ H_2 \\ S_2 \end{pmatrix} = \mathbf{f}_1 \tag{2.6}$$

where

$$D_2 = (n_1, n_2) \begin{pmatrix} [\rho_{xx}] & [\rho_{xy}] \\ [\rho_{yx}] & [\rho_{yy}] \end{pmatrix} \begin{pmatrix} n_1 \\ n_2 \end{pmatrix}, S_2 = (n_1, n_2) \begin{pmatrix} [p_{xx}] & [p_{xy}] \\ [p_{yx}] & [p_{yy}] \end{pmatrix} \begin{pmatrix} n_1 \\ n_2 \end{pmatrix},$$

$$H_2 = (n_1, n_2) \begin{pmatrix} u_2 \\ v_2 \end{pmatrix},$$

$$u_2 = (n_1, n_2) \begin{pmatrix} [u_{xx}] & [u_{xy}] \\ [u_{yx}] & [u_{yy}] \end{pmatrix} \begin{pmatrix} n_1 \\ n_2 \end{pmatrix}, v_2 = (n_1, n_2) \begin{pmatrix} [v_{xx}] & [v_{xy}] \\ [v_{yx}] & [v_{yy}] \end{pmatrix} \begin{pmatrix} n_1 \\ n_2 \end{pmatrix},$$

$$\mathbf{f}_1 = \begin{pmatrix} g^{-1} D_{0\xi}(h_0 - g^{-1} C_\xi) + D_1\left(2H_1 - \kappa H_0\right) + M_1 \\ g^{-1} H_{0\xi}(h_0 - g^{-1} C_\xi) + H_1^2 - D_1 S_1 (\rho_+ - D_0)^{-2} - g^{-1} h_0 C_\xi \\ g^{-1} S_{0\xi}(h_0 - g^{-1} C_\xi) + S_1\left((\gamma + 1) H_1 - \gamma \kappa H_0\right) + a_-^2 M_1 \end{pmatrix},$$

$$M_1 = (\rho_+ - D_0)\left\{\kappa^2 H_0 + \kappa H_1 - \frac{h_{0\xi}}{g}\right\} \tag{2.7}$$

The suffix ξ denotes differentiation with respect to the ray còordinate ξ , and $h_0 = g^{-1}(C+H_0)^{-1}\left(S_{0\xi}(\rho_+ - D_0)^{-1} - H_0 C_\xi\right)$.

3. First and second set of equations in shock ray theory

We collect here the equations required for the calculation of successive positions of a two-dimensional shock front and distribution of the shock strength (see ref [3] for details). The shock position is given by:

$$x_t = n_1 C, y_t = n_2 C \tag{3.1}$$

Let Θ be the angle made by the normal to the shock front Ω_t with the x-direction, then $n_1 = \cos\Theta, n_2 = \sin\Theta$. The evolution of Θ is given by:

$$\Theta_t = -\frac{C_\xi}{g} \tag{3.2}$$

The curvature κ of the shock front is related to Θ by the following relation:

$$\kappa = -\frac{\Theta_\xi}{g} \tag{3.3}$$

The metric g defined for the curve Ω_t (which in general is the metric tensor defined over the shock surface and it reduces to a scalar g in the present case, see ref [8]) evolves as:

$$g_t = -\kappa C g \tag{3.4}$$

The set of five equations(3. 1)-(3. 4) for x, y, Θ, κ and g is not closed, since the shock velocity depends upon the shock strength D_0. The evolution of D_0 and the other two quantities H_0 and S_0 is given by the equation (2. 5) . The set of six equations (3. 1)-(3. 4) and (2. 5), is again not complete due to presence of the quantities D_1, H_1, S_1 in the eqn (2. 5). The evolution of these quantities is given by the equation (2. 7). In a similar way, further compatibility conditions can be derived.

The shock ray theory consists of solving the infinite system of compatibility conditions along with the above system of five equations (3. 1)-(3. 4).

4. Reduction of vector compatibility conditions into scalar compatibility conditions and the new theory of shock dynamics

For computational convenience, it is easier to work with scalar form of compatibility conditions. The system of vector compatibility conditions can be reduced to that of scalar compatibility conditions by left multiplication of a suitably chosen eigenvector of the matrix $\mathbf{P}$ (see ref [5] for details). The eigenvalues of the matrix $\mathbf{P}$ are:

$$\lambda_1 = -(C + H_0); \lambda_{2,3} = -(C + H_0) \pm a_- \tag{4.1}$$

The left eigenvector corresponding to the eigenvalue λ_2, viz. $\mathbf{L} = (0, \frac{1}{2}(\rho_+ - D_0), \frac{1}{2}a_-^{-1})$ is chosen for left multiplication, and we introduce the following notations:

$$\pi_0 = D_0,$$

$$\pi_N = \mathbf{L}.U_N = \frac{1}{2}(\rho_+ - D_0)H_N + \frac{S_N}{2a_-} \tag{4.2}$$

where

$$\mathbf{U}_N = \begin{pmatrix} D_N \\ H_N \\ S_N \end{pmatrix}, N = 1, 2.....$$

Now, left-multiplying the equation (2. 5) with $\mathbf{L}$, we get

$$\pi_{0t} = -m\lambda_2\pi_1 - \frac{1}{2}ma_-C\pi_0\frac{\Theta_\xi}{g} \tag{4.3}$$

where

$$m = \left(\frac{1}{2}(\rho_+ - \pi_0)a + \frac{b}{2a_-}\right)^{-1}, a = \frac{C\rho_+(4\rho_+ + (\gamma-3)\pi_0)}{2(\rho_+ - \pi_0)^2(2\rho_+ + (\gamma-1)\pi_0)},$$

$$b = \frac{4\gamma p_+\rho_+}{(2\rho_+ + (\gamma-1)\pi_0)^2}$$

To express D_1, H_1, S_1 in terms of π_0 and π_1, we multiply eqn (2. 5) by $\mathbf{P}^{-1}$ to get

$$\mathbf{U}_1 = \begin{pmatrix} D_1 \\ H_1 \\ S_1 \end{pmatrix} = \mathbf{P}^{-1}\left\{-\frac{\Theta_\xi}{g}C\pi_0\begin{pmatrix} 1 \\ 0 \\ a_-^2 \end{pmatrix} + \begin{pmatrix} 1 \\ a \\ b \end{pmatrix}\left(m\lambda_2\pi_1 + \frac{1}{2}mCa_-\pi_0\frac{\Theta_\xi}{g}\right)\right\} \tag{4.4}$$

which gives

$$D_1 = d_{10}\pi_0\Theta_\xi + d_{11}\pi_1, H_1 = h_{10}\pi_0\Theta_\xi + h_{11}\pi_1, S_1 = s_{10}\pi_0\Theta_\xi + s_{11}\pi_1 \tag{4.5}$$

where

$$d_{10}=\frac{C}{g}\left\{\frac{1}{\lambda_1}(-1+\frac{ma_-}{2})-\frac{(\rho_+-\pi_0)}{2\lambda_2\lambda_3}ama_-+\frac{1}{\lambda_1\lambda_2\lambda_3}(-a_-^2+\frac{bma_-}{2})\right\},$$

$$d_{11}=m\left\{\frac{\lambda_2}{\lambda_1}-\rho_+-\pi_0)\frac{a}{\lambda_3}+\frac{b}{\lambda_1\lambda_3}\right\}$$

$$h_{10}=\frac{C}{g\lambda_2\lambda_3}\left\{\frac{ama_-\lambda_1}{2}-\frac{1}{(\rho_+-\pi_0)}(-a_-^2+\frac{bma_-}{2})\right\},h_{11}=\frac{m}{\lambda_3}\left(\lambda_1 a-\frac{b}{(\rho_+-D_0)}\right),$$

$$s_{10}=\frac{C}{g\lambda_2\lambda_3}\left(-\gamma(p_+-S_0)\frac{ama_-}{2}+\lambda_1(-a_-^2+\frac{bma_-}{2})\right),s_{11}=\frac{m}{\lambda_3}\left(-\gamma(p_+-S_0)+\lambda_1 b\right)$$

Repeating the same procedure of scalarisation with the second vector compatibility condition (viz. eqn (2. 7), we get after considerable simplifications:

$$\pi_{1t}=\alpha_1\pi_{0\xi}+\alpha_2\pi_{0\xi}^2+\alpha_3\pi_0^2\Theta_\xi^2+\alpha_4\pi_1^2+\alpha_5\pi_0\pi_1\Theta_\xi$$
$$+\alpha_6\pi_0\Theta_\xi^2+\alpha_7\pi_{0\xi\xi}+\alpha_8\pi_1\Theta_\xi+\alpha_9\pi_{0\xi}g_\xi-\lambda_2\pi_2 \tag{4.6}$$

where

$$\alpha_1=\frac{h_0}{2g}\left\{(a-\omega)(\rho_+-\pi_0)+\frac{b}{a_-}\right\},$$

$$\alpha_2=-\frac{1}{2g^2}\left\{\omega(\rho_+-\pi_0)+\frac{b\omega}{a_-}-\frac{a_-(\rho_+-\pi_0)}{\rho_+C}(\Delta+\frac{\omega(b-C\pi_0\omega)}{C})\right\},$$

$$\alpha_3=\frac{1}{2}(\rho_+-\pi_0)\left(h_{10}^2-\frac{d_{10}s_{10}}{(\rho_+-\pi_0)^2}\right)+\frac{s_{10}}{2a_-}\left((\gamma+1)h_{10}+\frac{\gamma C}{g(\rho_+-\pi_0)}\right)$$
$$+\frac{m}{4g}(h_{10}-es_{10})Ca_-,$$

$$\alpha_4=\frac{1}{2}(\rho_+-\pi_0)\left(h_{11}^2-\frac{d_{11}s_{11}}{(\rho_+-\pi_0)^2}\right)+\frac{(\gamma+1)}{2a_-}s_{11}h_{11}+\frac{1}{2}m\lambda_2(h_{11}-es_{11}),$$

$$\alpha_5=\frac{1}{2}(\rho_+-\pi_0)\left(2h_{10}h_{11}-\frac{(d_{10}s_{11}+d_{11}s_{10})}{(\rho_+-\pi_0)^2}\right)+\frac{(\gamma+1)}{2a_-}s_{10}h_{11}$$
$$+\frac{s_{11}}{2a_-}\left((\gamma+1)h_{10}+\frac{\gamma C}{g(\rho_+-\pi_0)}\right)+\frac{m}{2}(\lambda_2(h_{10}-es_{10})+\frac{Ca_-}{2g}(h_{11}-es_{11})),$$

$$\alpha_6=\frac{1}{2}a_-\left(\frac{C}{g^2}-(\rho_+-\pi_0)\frac{h_{10}}{g}\right),\alpha_7=-\frac{a_-(\rho_+-\pi_0)}{2\rho_+g^2C}(b-C\pi_0\omega),$$

$$\alpha_8=-\frac{a_-h_{11}}{2g}(\rho_+-\pi_0),\alpha_9=\frac{a_-(\rho_+-\pi_0)}{2\rho_+g^3C}(b-C\pi_0\omega),$$

and

$$\omega = \frac{\partial C}{\partial \pi_0} = -\frac{(\gamma+1)a_+^2\rho_+}{C(2\rho_+ + (\gamma-1)\pi_0)^2},$$

$$e = \frac{\partial}{\partial \pi_0}\Big(\frac{1}{a_-}\Big) = \frac{1}{2a_-(\rho_+ - \pi_0)}\Big(\frac{\gamma\rho_+^2 C^4}{a_+^2 a_-^2(\rho_+ - \pi_0)^2} - 1\Big),$$

$$\Delta = \Big(4(\gamma-1)\rho_+ + (\gamma^2-1)\pi_0\Big)\Lambda + C\omega, \Lambda = \frac{2a_+^2\rho_+}{\big(2\rho_+ + (\gamma-1)\pi_0\big)^3}$$

We note that all the coefficients are functions of π_0 only. The new theory of shock dynamics (NTSD) using the compatibility conditions upto the second is obtained by setting $\pi_2 = 0$ in the equation (4. 6). The equations (4. 3) and (4. 6) (with $\pi_2 = 0$), form a closed system of equation for the shock strength π_0 and a quantity π_1 (which is a linear combination of jumps in the first derivatives of flow variables). To calculate the successive positions of a two-dimensional shock front and distribution of shock strength π_0, the equations (4. 3) and (4. 6) (with $\pi_2 = 0$) have to be solved together with the equations (3. 1)-(3. 4).

A finite difference scheme using forward difference in t and central difference in ξ is suggested to solve the above system of the partial differential equations:

$$\frac{x_i^{n+1} - x_i^n}{\Delta t} = C_i^n \cos\Theta_i^n, \frac{y_i^{n+1} - x_i^n}{\Delta t} = C_i^n \sin\Theta_i^n,$$

$$\frac{\Theta_i^{n+1} - \Theta_i^n}{\Delta t} = -(g_i^n)^{-1}\Big(\frac{C_{i+1}^n - C_{i-1}^n}{2\Delta\xi}\Big), \frac{g_i^{n+1} - g_i^n}{\Delta t} = C_i^n\Big(\frac{\Theta_{i+1}^n - \Theta_{i-1}^n}{2\Delta\xi}\Big),$$

$$\frac{\pi_{0i}^{n+1} - \pi_{0i}^n}{\Delta t} = -m_i^n\Big\{\lambda_{2i}^n\pi_{1i}^n + \frac{1}{2}a_{-i}^n C_i^n\pi_{0i}^n\Big((g_i^n)^{-1}\big(\frac{\Theta_{i+1}^n - \Theta_{i-1}^n}{2\Delta\xi}\big)\Big)\Big\},$$

$$\frac{\pi_{1i}^{n+1} - \pi_{1i}^n}{\Delta t} = \alpha_{1i}^n\Big(\frac{\pi_{0i+1}^n - \pi_{0i-1}^n}{2\Delta\xi}\Big) + \alpha_{2i}^n\Big(\frac{\pi_{0i+1}^n - \pi_{0i-1}^n}{2\Delta\xi}\Big)^2$$

$$+\alpha_{3i}^n(\pi_{0i}^n)^2\Big(\frac{\Theta_{i+1}^n - \Theta_{i-1}^n}{2\Delta\xi}\Big)^2 + \alpha_{4i}^n(\pi_{1i}^n)^2 + \alpha_{5i}^n\pi_{0i}^n\pi_{1i}^n\Big(\frac{\Theta_{i+1}^n - \Theta_{i-1}^n}{2\Delta\xi}\Big)$$

$$+\alpha_{6i}^n\pi_{0i}^n\Big(\frac{\Theta_{i+1}^n - \Theta_{i-1}^n}{2\Delta\xi}\Big)^2 + \alpha_{7i}^n\Big(\frac{\pi_{0i+1}^n - 2\pi_{0i}^n + \pi_{0i-1}^n}{\Delta\xi^2}\Big)$$

$$+\alpha_{8i}^n\pi_{1i}^n\Big(\frac{\Theta_{i+1}^n - \Theta_{i-1}^n}{2\Delta\xi}\Big) + \alpha_{9i}^n\Big(\frac{\pi_{0i+1}^n - \pi_{0i-1}^n}{2\Delta\xi}\Big)\Big(\frac{g_{i+1}^n - g_{i-1}^n}{2\Delta\xi}\Big) \qquad (4.7)$$

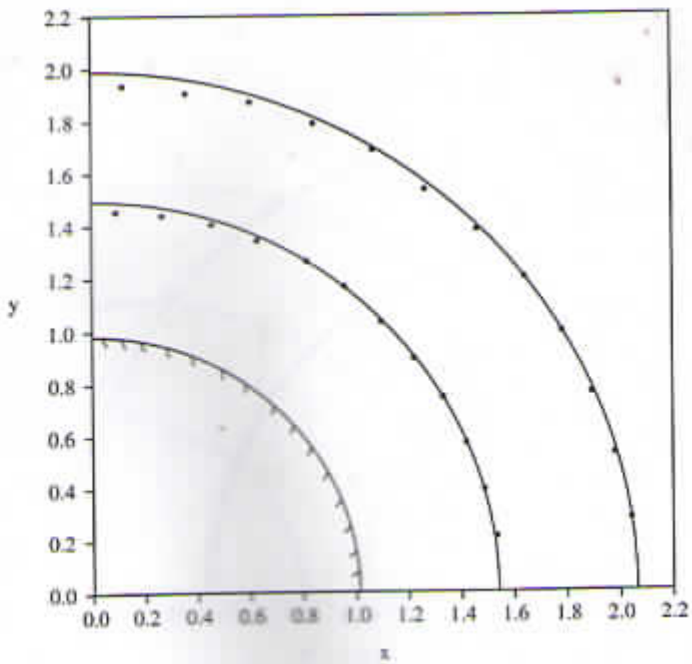

Figure 1.
Location and shape of a weak shock due to the motion of a circular cylinder moving with a velocity 0.10 at times 0.50 and 1.00 respectively. TTTT : blunt body; ——— : shock front by NTSD; – · – : shock front by Whitman's theory; · : shock front by FDM.

5. Solution of convex piston problem

The NTSD is now used to find the flow produced by a curved piston of elliptical shape with an arbitrarily assigned value of the eccentricity. We consider the flow produced by the curved piston which suddenly starts moving with a non-zero velocity into a gas at rest on its right side. Physically what happens under such conditions is that a shock is formed at the surface of the piston and moves outward. With increasing time, the shock asymptotically reaches its steady state position (see Hays and Probstein [11], p 445).

The flow variables are non-dimensionalised as follows:

$$x = L\bar{x}, y = L\bar{y}, t = \frac{L}{a_+}\bar{t}, u = a_+\bar{u}, v = a_+\bar{v}, C = a_+\bar{C}, \rho = \rho_+\bar{\rho}, p = \gamma p_+\bar{p} \quad (5.1)$$

where the overhead bar denotes the non-dimensional variable, L is a characteristic length which is chosen to be semi-major axis of the elliptic cross section.

We assume that the flow variables at a point P $(\bar{x}, \bar{y})$ behind the shock and in a small neighbourhood of $\bar{t} = 0$, can be expanded in a Taylor's series and let $(\bar{x}_0, \bar{y}_0)$

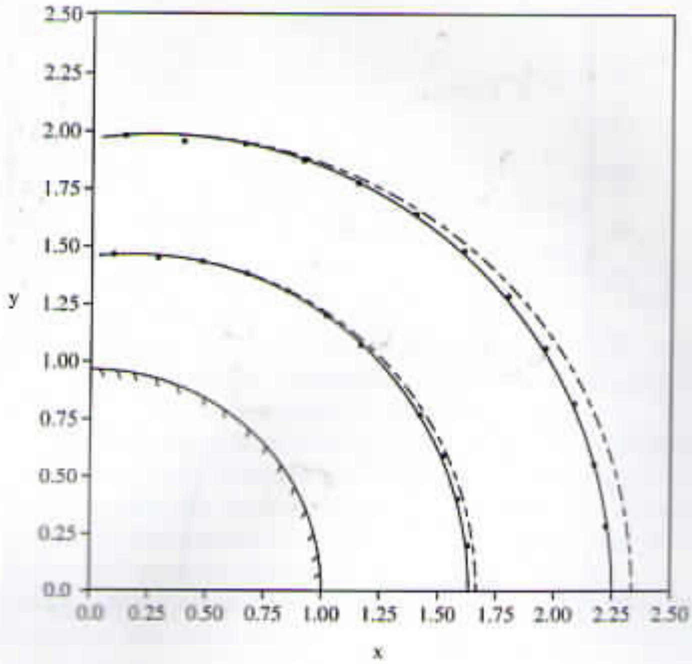

Figure 2.
Location and shape of a shock front due to the motion of an elliptic cylinder with eccentricity 0.25 and moving with a velocity 0.50 at times 0.50 and 1.00 respectively. TTTT : blunt body; ——— : shock front by NTSD; - - - : shock front by Whitman's theory; · : shock front by FDM.

(where $\bar{x}_0 = \bar{x}_0(\xi), \bar{y}_0 = \bar{y}_0(\xi)$) be the position of the point P on the piston at $\bar{t} = 0$:

$$\bar{\rho} = \rho_l + \rho_{11}(\bar{x} - \bar{x}_0) + \rho_{12}(\bar{y} - \bar{y}_0) + \rho_{13}\bar{t} + \dots\dots\dots$$
$$\bar{p} = p_l + p_{11}(\bar{x} - \bar{x}_0) + p_{12}(\bar{y} - \bar{y}_0) + p_{13}\bar{t} + \dots\dots$$
$$\bar{u} = u_l + u_{11}(\bar{x} - \bar{x}_0) + u_{12}(\bar{y} - \bar{y}_0) + u_{13}\bar{t} + \dots\dots\dots$$
$$\bar{v} = v_l + v_{11}(\bar{x} - \bar{x}_0) + v_{12}(\bar{y} - \bar{y}_0) + v_{13}\bar{t} + \dots\dots\dots \quad (5.2)$$

where ρ_l, p_l, u_l, v_l are the limiting values of the flow variables $\bar{\rho}, \bar{p}, \bar{u}$ and $\bar{v}$ at the point $(\bar{x}_0, \bar{y}_0)$ as we approach the shock front from the left at $\bar{t} = 0$.

The piston is assumed to move in x-direction only. Then its path can be expanded in the following Taylor's series in $\bar{t}$:

$$X_p(\xi, \bar{t}) = \bar{x}_0(\xi) + X_{p_1}\bar{t} + X_{p_2}\bar{t}^2 + \dots\dots, Y_p(\xi, \bar{t}) = \bar{y}_0(\xi) \quad (5.3)$$

Let (X,Y) be the co-ordinates of a point on the shock front, which can be expanded in the following form:

$$X(\xi, \bar{t}) = \bar{x}_0(\xi) + S_1^x(\xi)\bar{t} + S_2^x(\xi)\bar{t}^2 + \dots\dots, Y(\xi, \bar{t}) = \bar{y}_0(\xi) + S_1^y(\xi)\bar{t} + S_2^y(\xi)\bar{t}^2 + \dots\dots \quad (5.4)$$

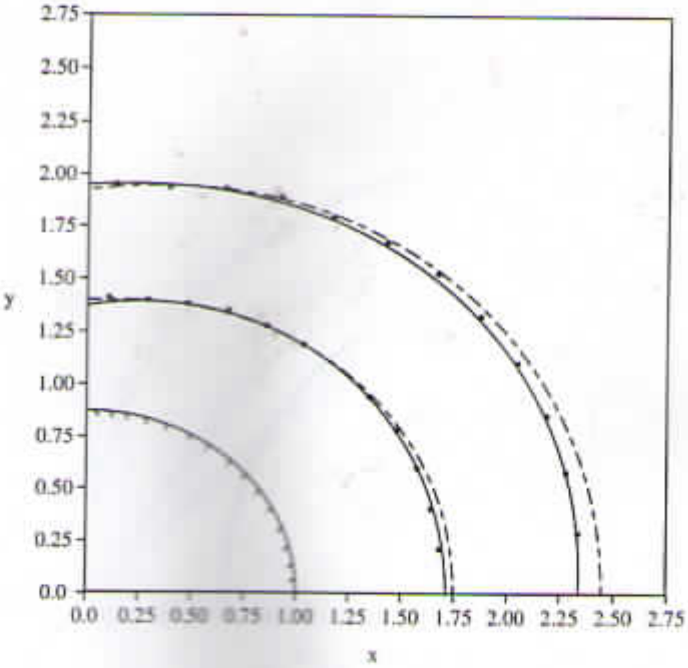

Figure 3.
Location and shock front due to the motion of an elliptic cylinder with eccentricity 0.50 and moving with a velocity 0.75 at times 0.50 and 1.00 respectively. TTTT : blunt body; ——— : shock front by NTSD; – · – : shock front by Whitman's theory; · : shock front by FDM.

The components of the piston velocity and the shock velocity (at a chosen point on the shock front) are obtained by differentiating the equations (5. 3) and (5. 4) respectively with respect to $\bar{t}$. Since the shock begins from the curved piston, we assume the shock to be coincident with the piston at $\bar{t}=0$. The shock speed $\bar{C}(\bar{t})$ at any time $\bar{t}$ is given by:

$$\bar{C} = C_0 + \frac{2}{C_0}(S_1^x S_2^x + S_1^y S_2^y)\bar{t} + \ldots.. \tag{5.5}$$

where $C_0 = \left((S_1^x)^2 + (S_1^y)^2\right)^{1/2}$ is the shock speed at $\bar{t}=0$.

At the shock front, the Rankine - Hugoniot conditions for the curved shocks are given as (Courant and Friedrichs [9], p 299):

$$\rho_-(N_- - C) = -\rho_+ C$$

$$\rho_- N_-(N_- - C) + p_- = p_+$$

$$\frac{1}{2}\rho_-(N_- - C)q_-^2 + \rho_-(N_- - C)e_- + N_- p_- = -\frac{\rho_+ C p_+}{\gamma - 1}$$

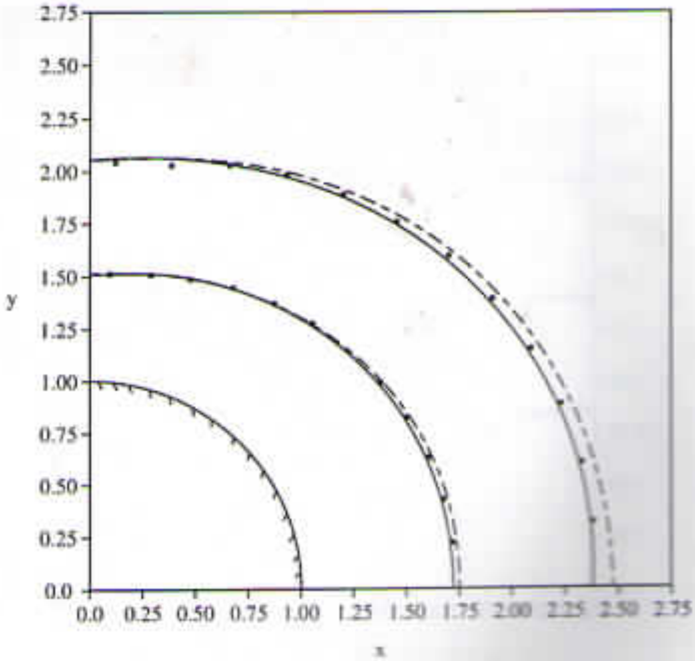

Figure 4.
Location and shape of a shock front due to the motion of a circular cylinder moving with a velocity 0.75 at times 0.50 and 1.00 respectively. ⊤⊤⊤⊤ : blunt body; —— : shock front by NTSD; - · - : shock front by Whitman's theory; · : shock front by FDM. (Comparison of Fig. 3 and 4 shows the effect of eccentricity on the motion of a curved shock.)

and the constancy of the tangential velocity component gives:

$$L_- = L_+ \tag{5.6}$$

where N is the normal velocity component$= u\cos\Theta + v\sin\Theta$, L is the tangential velocity component$=-u\sin\Theta + v\cos\Theta$, $q^2 = u^2 + v^2$ and e is the internal energy$=p/(\gamma-1)\rho$.

To obtain the initial conditions for the system of equations (4. 7), we require the coefficients in the Taylor's expansions (5. 2). For this purpose, we differentiate the relations (5. 2) with respect to $\bar{t}, \bar{x}$ and $\bar{y}$ and substitute into the equations of motion (2. 1)-(2. 3). Equating the coefficients of various powers of $\bar{t}, \bar{x}$ and $\bar{y}$, we get a system of linear equations for the coefficients $\rho_{11}, \rho_{12}, \rho_{13}$ etc. The coordinates (X, Y) of a chosen point on the shock front (as given by the eqn (5. 4)) are substituted into the Taylor expansions (5. 2), and finally we substitute these into the Rankine-Hugoniot conditions (5. 6) and equate the coefficients of various powers of $\bar{t}$ to obtain an additional system of linear equations.

The quantity C_0 is determined by the shock velocity at $\bar{t} = 0$ at a chosen point on the shock front when the curved piston begins to move with a non-zero velocity

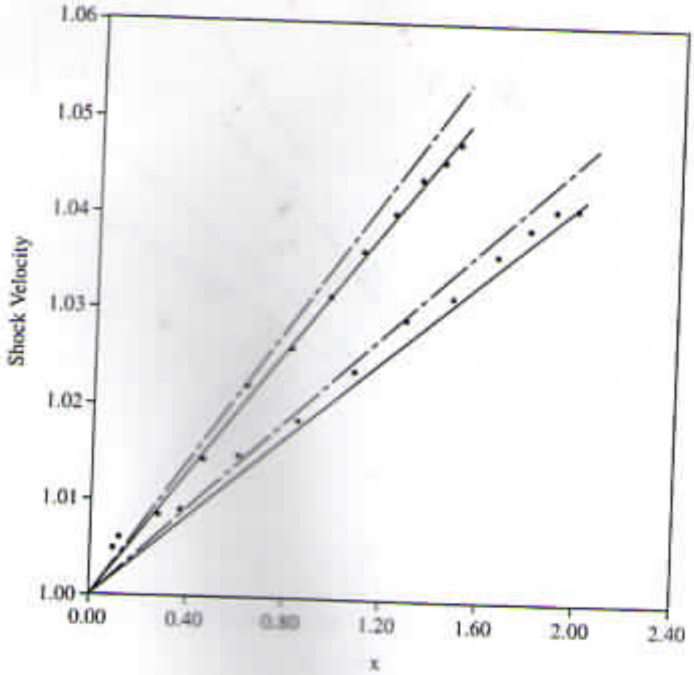

Figure 5.
Variation of shock strength along shock front for the case shown in Fig. 1

X_{p1}:

$$C_0 = \frac{X_{p1}\cos\Theta}{2(1-\mu^2)} + \left(1 + \frac{X_{p1}^{\ 2}\cos^2\Theta}{4(1-\mu^2)^2}\right)^{1/2} \tag{5.7}$$

where Θ is the angle made by the normal to the shock front with the x-axis at an arbitrarily chosen point on the shock front at $\bar{t} = 0$ and $\mu^2 = (\gamma - 1)/(\gamma + 1)$.

At the piston, the following boundary condition, namely, the fluid velocity at the piston equals the velocity of the piston, must be satisfied. Using this boundary condition, we get additional equations which enable us to determine all the unknowns in the equations (5. 2) and (5. 4) uniquely in terms of X_{p1}.

Finally, the initial conditions for π_0 and π_1 are obtained as:

$$\pi_0 = \frac{X_{p1}\cos\Theta}{X_{p1}\cos\Theta - C_0} \tag{5.8}$$

$$\pi_1 = \frac{1}{d_{11}}\left(\rho_N + g\kappa_0 d_{10}\pi_0\right) \tag{5.9}$$

where $\rho_N = (\rho_{11}\cos\Theta + \rho_{12}\sin\Theta)$, (where Θ can be expressed in terms of ξ), and κ_0 is the curvature of the shock front at $\bar{t} = 0$.

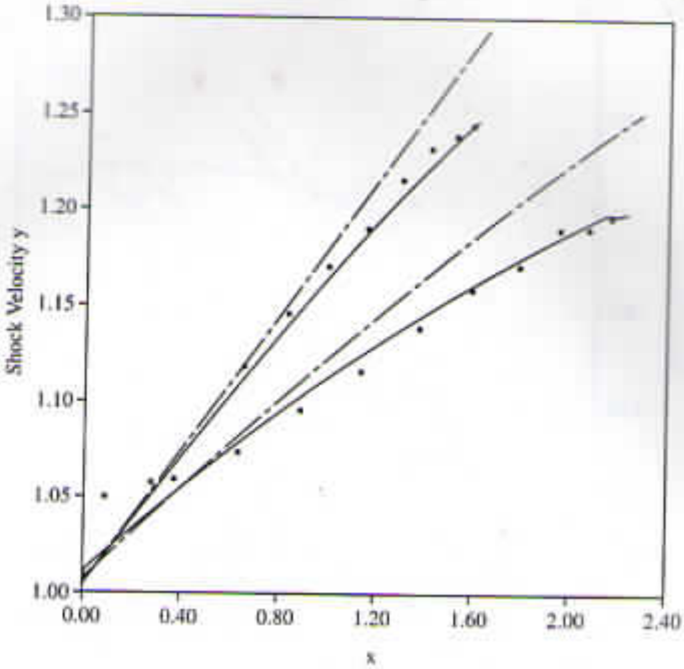

Figure 6.
Variation of shock strength along shock front for the case shown in Fig. 2

The system of finite difference equations (4. 7) can be solved now using the initial conditions as given above. A theoretical stability analysis for this finite difference scheme is not available at present. However, Prasad [10] has shown that NTSD equations are hyperbolic in (ξ, t) plane in the case of weak shocks. We have carried out extensive numerical experiments to obtain a practically stable value of $\Delta t/\Delta\xi$.

For illustration we have considered two-dimensional cylindrical blunt bodies of elliptic cross-sections with varying eccentricities.

6. Results and discussions

For numerical computations we take $\gamma = 1.4$. The shapes and the locations of the shock front have been computed by using the finite difference scheme (4. 7) for NTSD. The corresponding problems were also solved using Whitham's shock dynamics and also by using the flux-limited TVD version of MacCormack's scheme (FDM) (see Fletcher [20], Sweby [21] for details). Only the upper half of the shock profiles has been plotted due to symmetry about the x-axis.

In the fig. 1, the location and shape of a weak shock front, originating due to the

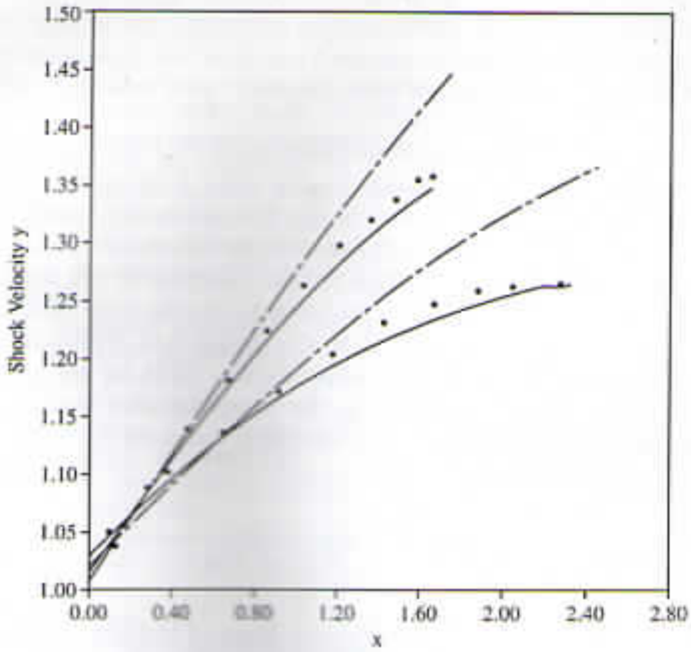

Figure 7.
Variation of shock strength along shock front for the case shown in Fig. 3

motion of a circular cylinder with a velocity $X_{p_1} = 0.10$ is shown at times $\bar{t} = 0.50$ and 1.0 respectively. We see that in this case, the locations and shapes of the shock front obtained by NTSD and Whitham's theory are almost the same. The results obtained by FDM also show very good agreement with those from NTSD, except for a few points near the y-axis. The variations of the shock strength along the shock front is shown in fig 5. The shock strengths in this case of a weak shock vary over a narrow range about the sonic limit.

The results for an elliptic body with eccentricity 0. 25, moving with a velocity 0. 50 is shown in the figures 2 and 6. We see in this case that whereas the NTSD and FDM predict almost the same locations for the shock front, the locations given by Whitham's theory are slightly ahead of those given by NTSD. An explanation of this behaviour is given by figure 6, where we see that the shock velocities given by the Whitham's theory at the points near the x-axis is appreciably higher than those by NTSD or FDM. The shock strengths predicted by NTSD and FDM are much closer, however they do not appear to follow any fixed pattern.

The results for an elliptic body with a velocity 0. 75 and eccentricity 0. 50 are shown in figures 3 and 7. The curves for the shock locations and shock strengths follow the same pattern as in the above case. We observe that the deviation in the shock strengths given by NTSD and Whitham's theory becomes appreciably large in this case. This deviation increases with an increase in the eccentricity, this fact becomes evident on comparing the figures 3 and 4, in which the shock shapes and locations for the elliptical body has been compared with those for a circular body moving with the same velocity.

We may conclude that the results obtained by NTSD are more accurate than those predicted by Whitham's classical theory, since it is capable of taking into account the effects of non-uniformities in the flow caused by continuous changes in the shape and strength of the shock front due to initial curvature of the convex shock front. This fact is also supported by the closeness of the results obtained by NTSD and those by FDM, which happens to be the only reliable approach at present to tackle such problems. It is also noted that NTSD consumes only nearly 3 % of the computational time taken by the FDM for the problem under consideration. This drastic reduction in the computational time occurs due to reduction in the number of independent variables, viz. from the (x, y, t) space (as in FDM) to (ξ, t) space (as in NTSD).

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