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Propagation of oblique shock waves in the troposphere

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Propagation of the shock waves, when propagated at a constant angle to the vertical, in the troposphere, is discussed. In a special case, the results of an earlier paper are deduced.

INTRODUCTION

Propagation of shock waves in the earth's atmosphere in the vertical direction was discussed by Singh (1969), in which the model of Mitra (1952) was used with minor changes. It was found that the shock velocity increases as the shock front propagates in the vertically upward direction. In the isothermal part of the atmosphere, the increase in shock velocity is smaller, but larger in layers where the temperature decreases. At a height of 78.44 km the fluid velocity behind the shock front becomes greater than the escape velocity. In this paper, we generalize the idea of previous work for oblique shock waves. Propagation of shock wave is considered only in the troposphere. Curvature of the earth is neglected, so that density, pressure and temperature of the atmosphere are varying in planes parallel to the surface of the earth. We consider the component of the shock front making angle θ to the vertical, so that the planes of variation of thermodynamical parameters are inclined at an angle θ to the shock surface which is assumed to be plane.

It is found that the rate of increase of shock velocity decreases as θ increases, being the maximum in vertical direction. In figure 2, the variation of shock velocity is shown for $\theta = 0$, $\pi/6$, $\pi/4$ as the shock propagates in the troposphere. In figure 3 the shock envelope at different intervals of time is drawn.

FORMULATION OF THE PROBLEM

If we take the point of explosion as origin and denote it by O, r_0 the radius vector, θ the angle made by the radius vector to the vertical and g_0 the acceleration due to gravity at a distance r_0 from the origin, then

$$g_0 = g_s \left(\frac{R_0}{R_0 + r_0 \cos\theta} \right)^2 \qquad \dots (1)$$

where g_s the value of g_0 at the surface of the earth, and R_0 the radius of the earth.

We assume that there is an explosion at the point O, which is on the surface of the earth. Shock wave created by the explosion propagates in all directions. We consider the component of the shock which propagates along the radius vector r_0 and inclined at an angle θ to the vertical. For simplicity we assume that the shock front is plane.

Let p_s, ρ_s, T_s, and p_0, ρ_0, T_0 be the pressure, density and absolute temperature at the surface of the earth and at a radial distance r_0, and $u, U, p, \rho, r, \bar{R}, t$, and g be the dimensionless fluid velocity, shock velocity, gas pressure, density, radius vector, earth's radius, time, and acceleration due to gravity, given by Singh (1969) in equation (4). Then equations of motion in terms of dimensionless parameters are given by

$$\left.\begin{aligned} &\frac{\partial \rho}{\partial t}+u\frac{\partial \rho}{\partial r}+\rho\frac{\partial u}{\partial r}=0 \\ &\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial r}+\frac{1}{\rho}\,\frac{\partial p}{\partial r}+\left(\frac{\bar{R}}{\bar{R}+r\cos\theta}\right)^2\cos\theta=0 \\ &\frac{\partial}{\partial t}(p\rho^{-\gamma})+u\frac{\partial}{\partial r}(p\rho^{-\gamma})=0 \end{aligned}\right\} \qquad \dots \quad (2)$$

In equilibrium conditions the variation of pressure in the atmosphere is given by

$$\frac{dp}{p}=-\left(\frac{\bar{R}}{\bar{R}+r\cos\theta}\right)^2\frac{\cos\theta}{T}\,dr \qquad \dots \quad (3)$$

where T is the dimensionless absolute temperature such that at the surface of the earth

$$p_s=R\rho_sT_s \qquad \dots \quad (4)$$

R being the gas constant. Here we have assumed that there is no wind in the troposphere.

When shock created at point O reaches the distance ξ, jump conditions across the shock front at $r=\xi$ are, (Pai, 1959, Singh 1969)

$$\left.\begin{aligned} \rho_2(u_2-U)&=-\rho U \\ p_2-p&=\rho u_2U \\ \frac{\gamma}{\gamma-1}\,\frac{p_2}{\rho_2}+\frac{(u_2-U)^2}{2}&=\frac{\gamma}{\gamma-1}\,\frac{p}{\rho}+\frac{U^2}{2} \end{aligned}\right\} \qquad \dots \quad (5)$$

where p_2, ρ_2, u_2 are the values of p, ρ, u behind the shock front and γ being adibatic gas constant. Quantities behind the shock front are functions of ξ and t whereas, ahead the shock front they are functions of ξ alone, ξ being the shock position. It is assumed that fluid velocity u in front of the shock is zero. Solving relations (5) for u_2, p_2, ρ_2 in terms of Mach number M, defined as

$$M=U/(\gamma\, p/\rho)^{\frac{1}{2}}$$

we get

$$\left.\begin{aligned} u_2(\xi, t) &= \frac{2c}{(\gamma+1)}\{M-M^{-1}\} \\ p_2(\xi, t) &= \frac{\gamma p}{(\gamma+1)} f(M) \\ \rho_2(\xi, t) &= \frac{(\gamma+1)\rho M^2}{g(M)} \end{aligned}\right\} \quad \dots \quad (6)$$

where

$$\left.\begin{aligned} f(M) &= \left\{ 2M^2 - \frac{\gamma-1}{\gamma} \right\} \\ g(M) &= \{2+(\gamma-1)M^2\} \\ c^2 &= \gamma p/\rho \end{aligned}\right\} \quad \dots \quad (7)$$

To visualize the problem we must know four unknown variables u_2, p_2, ρ_2 and M in terms of p, ρ. To relate these four variables we have three jump conditions (6). One extra relation between these four variables is required, which we get from the rule deviced by Whitham (1958). According to the rule, we write equations of motion just behind the shock front along the positive characteristic axis. In this equation we substitute the expressions for u_2, p_2, ρ_2 from (6).

Equation of motion along the positive characteristic axis, just behind the shock front is

$$dp_2+\rho_2 c_2 du_2+\frac{\rho_2 c_2}{u_2+c_2}\left(\frac{\bar{R}}{\bar{R}+\xi\cos\theta}\right)^2 \cos\theta \, d\xi = 0 \quad \dots \quad (8)$$

where ξ is the shock position. Using relations (6) in (8) and after some simplifications we get,

$$2\{(M^2+1)h(M)+2M^2\}\frac{dM}{M}+f(M)\frac{dp}{p}+2(M^2-1)h(M)\frac{dc}{c}$$

$$+\left(\frac{\bar{R}}{\bar{R}+\xi\cos\theta}\right)^2 \frac{(\gamma+1)^2 M^2 h(M)\cos\theta d\xi}{c^2\{2(M^2-1)+\sqrt{\gamma f g}\}} = 0 \quad \dots \quad (9)$$

where,
$$h(M) = \sqrt{\frac{\gamma f(M)}{g(M)}} \quad \dots \quad (10)$$

Relation (9) gives the variation of M, if p and c are known. If the variation of absolute temperature is known, the pressure can be found from (3) and thus M can be calculated from (9).

We discuss below the variation of M as the shock propagates in the troposphere. As the height of troposphere is very small compared to the radius of the earth, our assumption, that the earth's surface round the location is plane, is justified.

Shock waves in the troposphere

In the troposphere, which extends upto a height of 10 kilometers above the equator, the gases are in adiabatic equilibrium. In this region the variation of pressure and the equation of state in terms of dimensionless parameters p, ρ, can be writtes as

$$\frac{1}{\rho}\frac{dp}{dr} = -\left(\frac{\bar{R}}{\bar{R}+r\cos\theta}\right)^2 \cos\theta \qquad \ldots \text{(11)}$$

$$p = \rho^\gamma \qquad \ldots \text{(12}$$

Combining (11) and (12) we get after some simplifications

$$\frac{\gamma}{\gamma-1}\frac{d\rho^{\gamma-1}}{dr} = -\left(\frac{\bar{R}}{\bar{R}+r\cos\theta}\right)^2 \cos\theta \qquad \ldots \text{(13)}$$

Integrating (13) from the surface of the earth, where $r=0$, $\rho=1$ to the current point r, where density is ρ, we get,

$$\rho = \left[1 - \frac{\gamma-1}{\gamma}\,\frac{\bar{R}\; r\cos\theta}{\bar{R}+r\cos\theta}\right]^{\frac{1}{(\gamma-1)}} \qquad \ldots \text{(14a)}$$

From this we can easily get pressure p and temperature T as follows

$$p = \left[1 - \frac{\gamma-1}{\gamma}\,\frac{\bar{R}\, r\cos\theta}{\bar{R}+r\cos\theta}\right]^{\frac{\gamma}{\gamma-1}} \qquad \ldots \text{(14b)}$$

$$T = \left[1 - \frac{\gamma-1}{\gamma}\,\frac{\bar{R}\; r\cos\theta}{\bar{R}+r\cos\theta}\right] \qquad \ldots \text{(14c)}$$

Substituting the values of p and c at $r=\xi$ from equations (14) and (14b) in equation (9) after some simplifications we get

$$\frac{2}{M}\,\frac{dM}{d\xi} = K_3(M)\,\frac{\left(\dfrac{\bar{R}}{\bar{R}+\xi\cos\theta}\right)^2 \cos\theta}{\left[1 - \dfrac{\gamma-1}{\gamma}\,\dfrac{\bar{R}\,\xi\cos\theta}{\bar{R}+\xi\cos\theta}\right]} \qquad \ldots \text{(15)}$$

where

$$K_3(M) = K_1(M) + \frac{\gamma-1}{2\gamma}\,K_2(M). \qquad \ldots \text{(15a)}$$

$$K_1(M) = \frac{\left\{ f(M) - \frac{(\gamma+1)^2/\gamma M^2 h(M)}{2(M^2-1)+\sqrt{\gamma f g}} \right\}}{\{(M^2+1)h(M)+2M^2\}} \quad \ldots \quad (15b)$$

$$K_2(M) = \frac{2(M^2-1)h(M)}{\{(M^2+1)h(M)+2M^2\}} \quad \ldots \quad (15c)$$

In figure 1, we have drawn the value of $K_3(M)$ *versus* M, for $\gamma = 1.4$. It is found that the variation of $K_3(M)$ is small for $M \geqslant 3$, as compared to the variations of M.

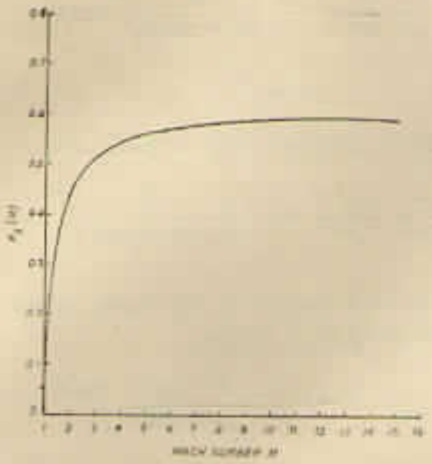

Figure 1. Variation of $K_3(M)$ *versus* M.

From figure 1, it is seen that $K_3(M)$ increases sharply as M increases from 1 to 3, having values 0 and 0.5156 at $M = 1$ and 3, respectively. When M increases from 3 onward, it's variation becomes negligible. We take the value of M at the surface of earth as equal to 4. Thus if we take $K_3(M)$, which no doubt is a function of M, as a constant and evaluate M from equation (15), the error is less than 2%. This error decreases as M increases. This fact can easily be seen from figure 1, where the curve showing the variation of $K_3(M)$, becomes a straight line as M becomes greater than 4.

Thus we neglect the variations in $K_3(M)$ in the process of integration of the equation (15). Integrating (15) and taking $K_3(M)$ as a constant during integration, we get,

$$M = M_s \left[1 - \frac{\gamma-1}{\gamma} \; \frac{\bar{R}\,\xi \cos\theta}{\bar{R}+\xi\cos\theta} \right]^{-\frac{\gamma}{\gamma-1}\frac{K_3}{2}} \quad \ldots \quad (16)$$

where M_s is the values of M at $\xi = 0$. From (16), (14) and the definition of M, we get shock velocity U, given by,

$$U = \sqrt{\gamma} M_s \left[1 - \frac{\gamma-1}{\gamma} \; \frac{\bar{R}\xi \cos\theta}{\bar{R}+\xi\cos\theta}\right]^{-\left(\frac{\gamma K_3}{\gamma-1}-1\right)/2} \qquad \ldots \quad (17)$$

Relations (16) and (17) give the variation of Mach number and the shock velocity as the shock propagates along the radial distance at an angle θ to the vertical. In figure 2, we have drawn the variation of shock velocity *versus* dimensionless

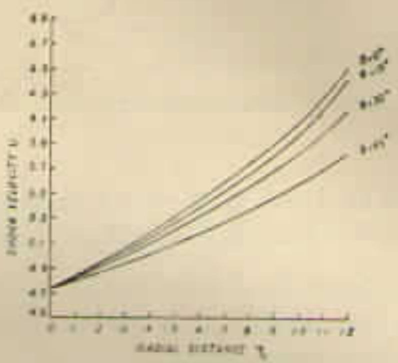

Figure 2. Variation of shock velocity U *versus* the radial distance ξ.

shock position ξ for $\theta = 0$, $\pi/6$, $\pi/4$. It is seen that the rate of increase of shock velocity decreases with the increase of θ.

Equation (17) can be written as,

$$\Delta\xi = \sqrt{\gamma} M_s \left[1 - \frac{\gamma-1}{\gamma} \; \frac{R\,\xi\cos\theta}{R+\xi\cos\theta}\right]^{-\left(\frac{\gamma K_3}{\gamma-1}-1\right)/2} \Delta t \qquad \ldots \quad (18)$$

where

$$\frac{d\xi}{dt} = U$$

From (18) we can compute the distance ξ at a particular time t. In figure 3, we have drawn the shock envelop at different time intervals after the explosion. All the figures are drawn in the dimensionless parameters.

Conclusion

In figure 2 we have plotted the variation of shock velocity in non-dimensional form. Here we have used the following data to nondimensionalize. $g_s = 981\ \mathrm{cm/sec^2}$, $p_s = 10^6\ \mathrm{dynes/cm^2}$, $\rho_s = 1.3\times10^{-3}\ \mathrm{gm/cm^3}$, $\alpha = 1275.3\times10^{-9}\ \mathrm{cm^{-1}}$ From computations we find that the shock velocity at a distance 9.4 km is 1.807,

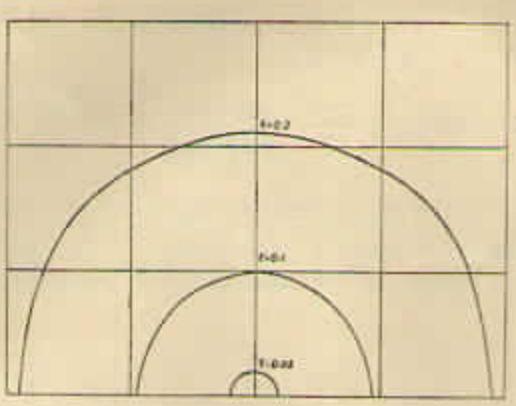

Figure 3. Position of shock front at dimensionless times 0.02, 0.1, 0.2 the origin being the point of explosion.

1.783, 1.712, 1.614 km/sec for $\theta = 0$, $\pi/12$, $\pi/6$ $\pi/4$, respectively. It is seen that the rate of increase of shock velocity is less for larger values of θ.

In figure 3 we have drawn the shock envelope at time intervals of 0.5655, 2.8275, 5.6550 seconds, respectively. It is shown that in a specific time the distance covered by the shock wave decreases as θ increases from 0 to $\pi/2$, being maximum for $\theta = 0$. In all the above calculations, the initial shock velocity is taken to be 1.313 km/sec.

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